Syntax & Semantics

Syntax		Semantics
symbols, signatures		structures
terms, formulae, sentences		their interpretations over structures
formal proof	—	**consequence**
syntactic derivation		semantic implication
derivability	—	**validity**
consistency	—	**satisfiability**
$\vdash$		$\models$

a toy example form mathematical practice

groups are structures of the form $\mathfrak{G} = (G, \circ^{\mathfrak{G}}, e^{\mathfrak{G}})$ satisfying

(G1) associativity of $\mathbf{e}^{\mathfrak{G}}$
(G2) $e^{\mathfrak{G}}$ right neutral for $\circ^{\mathfrak{G}}$
(G3) existence of right inverses for $\circ^{\mathfrak{G}}$

some **provable consequences:**

(Thm1) every right inverse is a left inverse
(Thm2) the right neutral is a left neutral
(Thm3) uniqueness of neutral element
(Thm4) uniqueness of inverses

σ-structures and basic universal algebra

$$\mathfrak{A} = \Big(A, (f^{\mathfrak{A}})_{c \in \mathrm{fctn}(\sigma)}, (R^{\mathfrak{A}})_{c \in \mathrm{rel}(\sigma)}, (c^{\mathfrak{A}})_{c \in \mathrm{const}(\sigma)}\Big)$$

with universe/domain $A \neq \emptyset$

and interpretations

$\mathfrak{I}^{\mathfrak{A}}(f) = f^{\mathfrak{A}} \colon A^n \to A$ for n-ary function symbol f

$\mathfrak{I}^{\mathfrak{A}}(R) = R^{\mathfrak{A}} \subseteq A^n$ for n-ary relation symbol R

$\mathfrak{I}^{\mathfrak{A}}(c) = c^{\mathfrak{A}} \in A$ for constant symbol c

supporting natural notions from universal algebra:

substructures/expansions
reducts/expansions
homomorphisms, isomorphisms, automorphisms, ...

syntax: σ-terms

inductive definition:

T_σ, the set of σ-terms (over Var), is the smallest set s.t.

(i) $\mathrm{Var} \subseteq T_\sigma$

(ii) $\mathrm{const}(\sigma) \subseteq T_\sigma$

(iii) for n-ary $f \in \mathrm{fctn}(\sigma)$: if $t_1, \ldots, t_n \in T_\sigma$, then $f t_1 \ldots t_n \in T_\sigma$

calculus:

(T1) $\dfrac{}{x}$ $x \in \mathrm{Var}$

(T2) $\dfrac{}{c}$ $c \in \mathrm{const}(\sigma)$

(T3) $\dfrac{t_1, \ldots, t_n}{f t_1 \ldots t_n}$ $f \in \mathrm{fctn}(\sigma)$ n-ary

σ-terms and free term structures

$$\mathfrak{T}_\sigma = \left(T_\sigma, \mathfrak{J}^{\mathfrak{T}_\sigma}\right)$$

a $(\mathrm{const}(\sigma) \,\dot\cup\, \mathrm{fctn}(\sigma))$-structure where

for $c \in \mathrm{cons}(\sigma)$:

$\mathfrak{J}^{\mathfrak{T}_\sigma}(c) =^{\mathfrak{T}_\sigma} (c) := c$

for n-ary $f \in \mathrm{fctn}(\sigma)$:

$$\mathfrak{J}^{\mathfrak{T}_\sigma}(f) := f^{\mathfrak{T}_\sigma}: \qquad \begin{array}{rcl} f^{\mathfrak{T}_\sigma} : (T_\sigma)^n & \longrightarrow & T_\sigma \\ (t_1, \dots, t_n) & \longmapsto & f\,t_1 \dots t_n \end{array}$$

$\rightarrow$ interpretations of σ-terms in σ-structures via homomorphisms

$\mathrm{FO}(\sigma)$: σ-formulae

calculus:

(F1)	$\dfrac{}{t{=}t'}$	$t, t' \in T_\sigma$	atomic formulae
(F2)	$\dfrac{}{Rt_1 \ldots t_n}$	$R \in \mathrm{rel}(\sigma)$ n-ary, $t_i \in T_\sigma$	
(F3)	$\dfrac{\varphi}{\neg\varphi}$		boolean connectives
(F4)	$\dfrac{\varphi_1\,,\varphi_2}{(\,\varphi_1 * \varphi_2\,)}$	$* = \wedge, \vee, \rightarrow, \leftrightarrow$	
(F5)	$\dfrac{\varphi}{Qx\varphi}$	$x \in \mathsf{Var}$, $Q = \forall, \exists$	quantification

syntactic induction

. . . refers to induction w.r.t. rules of some calculus (inductive process of generation) and

- proves an assertion $A(o)$
  for all objects o generated by the calculus, or
- defines new objects/entities/relationships . . .
  for all objects o generated by the calculus

> e.g., to prove $A(o)$ for all o, show
> that each rule $\frac{o_1, \ldots, o_m}{o}$ of the calculus
> is such that $A(o_1), \ldots A(o_m)$ implies $A(o)$

idea: very low-level finitistic combinatorics
should suffice at this 'foundational' stage

examples of proofs/definitions by syntactic induction

on terms $\mathbf{t} \in \mathbf{T}_\sigma$:

unique parsing for all $t \in T_\sigma$, $\mathrm{var}\colon T_\sigma \to \mathcal{P}(\mathrm{Var})$, . . .

> interpretation map $\mathfrak{I}^{\mathfrak{A}}_{\beta}\colon T_\sigma \to A$

on formulae $\varphi \in \mathrm{FO}(\sigma)$:

unique parsing for all $\varphi \in \mathrm{FO}(\sigma)$, $\mathrm{free}\colon \mathrm{FO}(\sigma) \to \mathcal{P}(\mathrm{Var})$, . . .

> satisfaction relation $\mathfrak{A}, \beta \models \varphi$

assignments, and interpretation of terms

assignment $\beta\colon \mathrm{Var} \rightarrow A$

(temporary) interpretation of variables (as if constants)

σ-interpretation $\mathfrak{I}^{\mathfrak{A}}_{\beta}$ (or $(\mathfrak{A}, \beta) = (A, \mathfrak{I}^{\mathfrak{A}}, \beta) = (A, \mathfrak{I}^{\mathfrak{A}}_{\beta})$)

interpretation of all symbols in σ and all $x \in \mathrm{Var}$

interpretation of terms $t \in T_\sigma$
over σ-interpretation $\mathfrak{I} = \mathfrak{I}^{\mathfrak{A}}_{\beta}$:

$$\begin{array}{rcl} \mathfrak{I}\colon T_\sigma & \longrightarrow & A \\ t & \longmapsto & \mathfrak{I}(t) = t^{\mathfrak{I}} \end{array}$$

the natural extension of $\mathfrak{I} \restriction (\mathrm{const}(\sigma) \cup \mathrm{Var})$
to a homomorphism !

interpretation of formulae: satisfaction relation

satisfaction relation $\models$

between σ-interpretations $\mathfrak{I} = \mathfrak{I}^{\mathfrak{A}}_{\beta} = (\mathfrak{A}, \beta)$ and $\varphi \in \mathrm{FO}(\sigma)$

defined by syntactic induction over φ,
for fixed $\mathfrak{A}$ and all assignments β simultaneously:

(F1) $(\mathfrak{A}, \beta) \models t = t'$ if $\mathfrak{I}^{\mathfrak{A}}_{\beta}(t) = \mathfrak{I}^{\mathfrak{A}}_{\beta}(t')$

(F2) $(\mathfrak{A}, \beta) \models Rt_1 \dots t_n$ if $(\mathfrak{I}^{\mathfrak{A}}_{\beta}(t_1), \dots, \mathfrak{I}^{\mathfrak{A}}_{\beta}(t_n)) \in R^{\mathfrak{A}}$

(F3/4) ... the obvious extensional boolean clauses

(F5) $(\mathfrak{A}, \beta) \models \exists x \varphi$ if $(\mathfrak{A}, \beta\frac{a}{x}) \models \varphi$ for some $a \in A$

$(\mathfrak{A}, \beta) \models \forall x \varphi$ if $(\mathfrak{A}, \beta\frac{a}{x}) \models \varphi$ for all $a \in A$