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Excursion: well-orderings, ordinals, Zorn's lemma and the axiom of choice (AC).

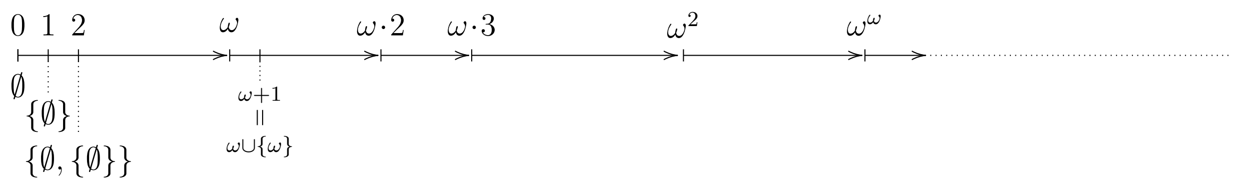
Set theory concerns the "universe of all sets" (not itself a set, hence not a structure in our usual sense) with the $\in$-relation. The natural numbers can be represented in set theory by

$$\begin{aligned} 0 &:= \emptyset, \\ 1 &:= \{\emptyset\}, \\ 2 &:= \{\emptyset, \{\emptyset\}\}, \\ 3 &:= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\ &\vdots \\ n+1 &:= n \cup \{n\} \end{aligned}$$

One checks (e.g., by induction) that each one of these finite sets is well-ordered by the $\in$-relation, and has all elements of its elements as elements, i.e., every element is a subset (such sets are called *transitive*). *Ordinals* are transitive sets that are well-ordered by the $\in$-relation. Natural numbers as well as the set ω of all natural numbers are ordinals. Ordinals generalise the order-theoretic, inductive properties of the natural numbers into the transfinite. One can show that for any two ordinals, one is an initial segment of the other (any set of ordinals forms a chain w.r.t. the relation of being a "proper end extension"). The union of any chain of ordinals is again an ordinal; and if the chain had no maximal element, then the resulting ordinal is greater than every member of the chain and referred to as a *limit ordinal.* The set ω of natural numbers is the first such limit: the union of the chain of all its predecessors, i.e., of all the finite ordinals. But then the successor of that set, $\omega \cup \{\omega\}$, and its successor, $\omega \cup \{\omega, \omega \cup \{\omega\}\}$, etc. continue the sequence.

The class (not set) of all ordinals, denoted On, is itself transitive (has every element of its elements as an element) and the $\in$-relation linearly well-orders On in the sense that it behaves like a linear ordering and that every non-empty sub-class has an $\in$-minimal member. On is closed under the successor operation that maps an ordinal α to its immediate successor $\alpha + 1 := \alpha \cup \{\alpha\}$ (adding one new point, itself labelled α, to the ordering of α), and under taking limits, i.e., unions of chains of (smaller) ordinals.

In set theory (ZF) one can show that there are arbitrarily large ordinals, also in the sense that On cannot be injectively mapped into any set (hence it cannot itself be a set, but must be a proper 'class').



Many nice features of the familiar inductive order type of the natural numbers extend to all other ordinals and to the class On. Here are two examples concerning, respectively, proofs and definitions by (transfinite) induction or recursion.

Transfinite Induction. *Let A be any subclass of On (the class of those ordinals α satisfying some given property $A(\alpha)$). If A contains 0 and is is closed under successor and limits (unions of chains of ordinals),[1] then $A = \mathsf{On}$, i.e., $A(\alpha)$ for all ordinals α.*

Transfinite Recursion. *Let g be any operation on sets. Then there is a unique operation F on On such that for every ordinal α, $F(\alpha) = g(F\restriction\alpha)$.*[2]

Exercise 1 [well-orderings, this is also Ex 7.1]
Consider the signature $\sigma = \{<\}$ with a binary relation symbol $<$. A σ-structure $\mathfrak{A} = (A, <^{\mathfrak{A}})$ is called a *well-ordering* (or $<^{\mathfrak{A}}$ is said to well-order A) if $<^{\mathfrak{A}}$ is a strict linear ordering of the universe A such that every non-empty subset $A' \subseteq A$ has a $<^{\mathfrak{A}}$-minimal element.

(a) Show that a linear ordering $\mathfrak{A} = (A, <^{\mathfrak{A}})$ is a well-ordering if, and only if, there is no infinite descending sequence w.r.t. $<^{\mathfrak{A}}$.

(b) Show that the class of all well-orderings is not Δ-elementary.

Exercise 2 [Zorn's lemma: this was already an extra exercise, Ex 6.4]
Apply Zorn's lemma to suitable partial orderings in order to prove the following:

(a) The *axiom of choice*: for every family $(A_i)_{i\in I}$ of non-empty sets A_i (indexed by any set I), there is a choice function $f\colon I \to \bigcup_{i\in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$.

(b) The *well-ordering principle*: every set A can be well-ordered, i.e., there exists a binary relation $<^{\mathfrak{A}} \subseteq A \times A$ such that $(A, <^{\mathfrak{A}})$ is a well-ordering (cf. Exercise 1 above).

Hint: think of partially ordered sets of suitable 'approximations' to the desired object, such that maximality of an approximation means that it is as desired. Also compare the exercise below: Zorn's lemma is in fact equivalent to both the above.

Exercise 3 [for lovers of AC/Zorn/ordinals]
Sketch proofs that the following are equivalent:

(i) The well-ordering principle: every set can be well-ordered.

(ii) The axiom of choice (AC): there is a choice function for every family of non-empty sets.

(iii) Zorn's lemma.

Hints: Using (i), choices for a choice function as required in (ii) can be based on selecting minimal elements w.r.t. a given well-ordering. Such definable choice functions do not rely on (AC). Into an inductive partial ordering *without* maximal elements, one could recursively define an order-preserving embedding of On onto a chain in the given partial ordering. For this select always some strict upper bound of the set of all previous values: this choice can be based on an arbitrary well-ordering of the domain of the partial order if we use (i), or on a suitable choice function on its power set if we use (ii). This contradicts the fact that, as a proper class, On is too large to fit into any set.

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[1] This is equivalently (!) summed up as: if for all $\alpha \in \mathsf{On}$, $A(\beta)$ for all $\beta \in \alpha$ implies $A(\alpha)$.

[2] $F\restriction\alpha$ stands for the restriction of F to arguments from α, i.e., to all smaller ordinals $\beta \in \alpha$. In set theoretic terms, this restriction is formalised by its graph $\{(\beta, F(\beta))\colon \beta \in \alpha\}$.