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Solution Hints for Exercises No.8

Exercise 1 (i) Consider $(\mathbb{N}, <)$ the structure of the ordered natural numbers and replace it with $(\mathbb{N}, E)$, where $nEm :\Leftrightarrow (n < m \vee m < n)$. This constitutes an infinite connected graph. Do the same with $(\mathbb{N}', <)$ a non-standard model for (the axioms of) the naturals. This $(\mathbb{N}', E)$ constitutes an infinite disconnected graph, since there is not finite chain $0Ex_0Ex_1E \dots Ex_kEn$ when n is an infinite non-standard natural (otherwise n would be finite). Recall that $(\mathbb{N}, <)$ and $(\mathbb{N}', <)$ are elementary equivalent, and so are $(\mathbb{N}, E)$ and $(\mathbb{N}', E)$ for similar reasons. Now assume that the class of the connected graphs were Δ-elementary, described by a set of sentences Φ. So $(\mathbb{N}, E)$, as a connected graph, would be a model for it. Then $(\mathbb{N}', E)$, as elementary equivalent to $(\mathbb{N}, E)$, would also be a model for Φ, contradiction since $(\mathbb{N}', E)$ is disconnected. Similarly, the class of the disconnected graphs is not Δ-elementary.

(ii) Similar to (i), use the ordered field of the real numbers and a corresponding non-standard model.

(iii) An abelian group $(G, +)$ is divisible if for all positive $n \in \mathbb{N}$ and $x \in G$, there exists $y \in G$ such that $x = ny$. For all positive n, let $\varphi_n := \forall x \exists y (x = ny)$. The abelian group axioms together with the set of formulas $\{\varphi_n \mid n \in \mathbb{N} - \{0\}\}$ characterise the divisible groups exactly, which are therefore Δ-elementary.

Exercise 2 (i) $\Rightarrow$ (ii). Let $0^{\mathfrak{A}} \in X$, $X \subseteq A$ closed under $\mathsf{S}^{\mathfrak{A}}$. Suppose $X \subsetneq A$. Then there is, by (i), a minimal $a \in A \setminus X$. $a \neq 0^{\mathfrak{A}}$ as $0^{\mathfrak{A}} \in X$ by assumption, so $a = \mathsf{S}^{\mathfrak{A}}(b)$ for some $b \in A$ ($\text{image}(\mathsf{S}^{\mathfrak{A}}) = A \setminus \{0^{\mathfrak{A}}\}$ by assumption). Since $\mathsf{S}^{\mathfrak{A}}(b) = a$ implies $b <^{\mathfrak{A}} a$, minimality of a implies that $b \in X$. But $b \in X$ and $a = \mathsf{S}^{\mathfrak{A}}(b) \notin X$ is impossible, as X is closed under $\mathsf{S}^{\mathfrak{A}}$.

(ii) $\Rightarrow$ (iii). Let $X \subseteq A$ be the set of those $a \in A$ for which $[0, a]^{\mathfrak{A}} := \{b \in A \colon b <^{\mathfrak{A}} a \text{ or } b = a\}$ is finite. Clearly $0 \in X$: $[0, 0]^{\mathfrak{A}} = \{0^{\mathfrak{A}}\}$ as $0^{\mathfrak{A}}$ is $<^{\mathfrak{A}}$-minimal. X is also closed under $\mathsf{S}^{\mathfrak{A}}$: $[0, \mathsf{S}^{\mathfrak{A}}(a)]^{\mathfrak{A}} = [0, a]^{\mathfrak{A}} \cup \{\mathsf{S}^{\mathfrak{A}}(a)\}$ as $\mathsf{S}^{\mathfrak{A}}(a)$ is the immediate successor of a. Hence $X = A$ by (ii).

(iii) $\Rightarrow$ (iv). Define $h \colon A \to \mathbb{N}$ through $h(a) := |[0, a]^{\mathfrak{A}}| - 1$. One shows that h is a bijection (injectivity follows as $a <^{\mathfrak{A}} a'$ implies $[0, a]^{\mathfrak{A}} \subsetneq [0, a']^{\mathfrak{A}}$) and that $h(\mathsf{S}^{\mathfrak{A}}(a)) = h(a) + 1$. Compatibility of h with $<$ and 0 are easily checked.

(iv) $\Rightarrow$ (i),(ii),(iii) are obvious.

Exercise 3 We write S for the successor function that maps $x \in \omega$ to $\mathsf{S}x := x \cup \{x\}$ (all operations and defined notions in terms of a model of ZFC – we work "within ZFC"). A set a is inductive if $\emptyset \in a$ and a is closed under S.

For (a) show that the subset $a := \{x \in \omega \colon \text{"}x \text{ transitive"}\} \subseteq$ is inductive, hence equal to ω.

$\emptyset \in a$: $\emptyset$ is transitive (as it has no elements).

Closure under S: let $z \in \omega$ be transitive. Consider $y \in \mathsf{S}z = z \cup \{z\}$, we need to show that $y \subseteq \mathsf{S}z$. If $y \in z$, then by transitivity of z we get $y \subseteq z$, whence $y \subseteq z \cup \{z\} = \mathsf{S}z$; if $y = z$, then $y = z \subseteq \mathsf{S}z$ is trivial.

Transitivity of $\in$ over ω: if $x \in y \in z \in \omega$, then transitivity of z implies that $y \subseteq z$, whence $x \in z$ follows.

For (b) similarly show that $a := \{x \in \omega \colon \neg\, x \in x\}$ is inductive.

$\emptyset \in a$: $\emptyset \notin \emptyset$.

Closure under S: let $z \in \omega$, $z \notin z$. Assume $\mathsf{S}z \in \mathsf{S}z$. Then $z \cup \{z\} = z$ (which would imply $\{z\} \subseteq z$ and thus also $z \in z$) or $z \cup \{z\} \in z$ (which by transitivity of z would imply $z \cup \{z\} \subseteq z$ and thus $z \in z$); hence both cases contradict the inductive hypothesis that $z \notin z$.

For (c) we show that $a := \{x \in \omega \colon \forall y(y \in \omega \to (x \in y \vee x = y \vee y \in x))\}$ is inductive.

For $\emptyset \in a$, show that the set $b := \{x \in \omega \colon x = \emptyset \vee \emptyset \in x\}$ is inductive (this is easy).

Closure of a under S: assume that $\forall y(y \in \omega \to (z \in y \vee z = y \vee y \in z))$ and show that

$$\forall y\big(y \in \omega \to (\mathsf{S}z \in y \vee \mathsf{S}z = y \vee y \in \mathsf{S}z)\big).$$

Consider some $y \in \omega$. From the assumption, we have $z \in y \vee z = y \vee y \in z$. If $y \in z$ or $y = z$, then $y \in z \cup \{z\} = \mathsf{S}z$. To cover the case of $z \in y$ we show by induction that for all $z, y \in \omega$:

$$z \in y \to \big(\mathsf{S}z = y \vee \mathsf{S}z \in y\big).$$

For this we show that the set $b := \{y \in \omega \colon z \in y \to (\mathsf{S}z = y \vee \mathsf{S}z \in y)\}$ is inductive:

$\emptyset \in b$ is trivial; b is closed under S: let $y \in b$ and assume $z \in \mathsf{S}y = y \cup \{y\}$. Then $z = y$ and $\mathsf{S}y = \mathsf{S}z$, or $z \in y$ and hence (as $y \in b$) $\mathsf{S}z = \mathsf{S}y$, or $\mathsf{S}z \in y \in \mathsf{S}y$ and $\mathsf{S}z \in \mathsf{S}y$ by transitivity. So in both cases $y \cup \{y\} \in b$.

Exercise 4

(a) Assuming $x_0 \in x_1 \in \cdots \in x_{n-1} \in x_0$, repeated application of (PAIR) and (union) yields the set $x = \{x_0, \ldots, x_{n-1}\}$. [For every i, we get $\{x_i\} = \{x_i, x_i\}$ by (PAIR), then inductively $\{x_0, \ldots, x_i, x_{i+1}\} = \{x_0, \ldots, x_i\} \cup \{x_{i+1}\}$ by (union).]
Now show that x violates (FOUND): each element $x_i \in x$ has non-empty intersection with x: for $i = 0$, $x_{n-1} \in x_i \cap x$; for $0 < i < n$, $x_{i-1} \in x_i \cap x$.
For injectivity of $x \mapsto x \cup \{x\}$: suppose $x \cup \{x\} = y \cup \{y\}$ and $x \neq y$; then $x \in y \cup \{y\}$ implies $x \in y$ and $y \in x \cup \{x\}$ implies $y \in x$, contradiction.

(b) Assume that $\neg\varphi(x)$ for some transitive x. Consider $w := \{z \in x \colon \neg\varphi(z)\} \subseteq x$.
By the assumption about φ, $w \neq \emptyset$. By (FOUND), the set w has an $\in$-minimal member u for which:
– $u \in x$ and $\neg\varphi(u)$ (as $u \in w$);
– $u \cap w = \emptyset$ ($\in$-minimality in w).
By transitivity of x, $u \in x$ implies $u \subseteq x$. As $u \subseteq x$, $u \cap w = \emptyset$ implies $\forall z(z \in u \to \varphi(z))$. This, however, implies $\varphi(u)$ by the assumption about φ, a contradiction.

Exercise 5

(a) $\forall x(x \neq \emptyset \wedge \forall z(z \in x \to z \neq \emptyset) \to \exists y(\forall z(z \in y \leftrightarrow \varphi(z, y))))$ where $\varphi(z, y)$ says that z is the graph of a function f from x to $\bigcup x$ such that $f(u) \in u$ for all $u \in x$.
Non-emptiness of the product is easily stated, and can be shown on the basis of (AC): look at a choice set for the set $\{\{u\} \times u \colon u \in x\}$.

(b) for all x, no function is an injection of $P(x)$ into x, or equivalently (why?): no function is a surjection from x onto $P(x)$. Follows (in ZFC) from the usual diagonalisation trick: assuming $f\colon x \to P(x)$ is a surjection, find that $d = \{u \in x\colon u \notin f(u)\} \subseteq x$ cannot be in the image of f, since $d = f(u)$ implies $u \in d$ iff $u \notin f(u) = d$.

(c) The set of all sequences of natural numbers is the set of all functions from ω to ω; each such function is a subset of $\omega \times \omega$, hence an element of $P(P(P(\omega)))$, so the set (!) of all natural number sequences can be isolated by separation as a subset of $P(P(P(\omega)))$. That it is not countable means there is no surjection from ω onto this set. This follows by diagonalisation (as usual) or as a special case of (b), since $P(\omega)$ is bijectively related to the set of natural sequences with range $2 = \{0, 1\}$ (characteristic functions).