Introduction to Mathematical Logic

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Solution Hints for Exercises No.7

Exercise 1

(a) Clearly the members of any infinite descending sequence $a_0 > a_1 > a_2 > \cdots$ form a set without minimal element. Conversely, any subset $A' \subseteq A$ without minimal element would allow us to choose inductively a descending sequence $(a_i)_{i\in\mathbb{N}}$ in A': since $a_i \in A'$ is not minmal, there is some $a_{i+1} \in A'$ for which $a_{i+1} <^{\mathfrak{A}} a_i$.

(b) Assume $\Phi \subseteq \mathrm{FO}(\{<\})$ axiomatised the class $\mathcal{WO}$. Consider $\Psi := \Phi \cup \{v_{n+1} < v_n \colon n \in \mathbb{N}\}$. Any finite subset of Ψ is satisfiable in a sufficiently large finite (well-)ordering, hence consistent. So Ψ as a whole is consistent, and therefore satisfiable by the Completeness Theorem. But no model $\mathfrak{A}, \beta \models \Psi$ can be well-ordered, as $(\beta(v_i))_{i\in\mathbb{N}}$ is a descending sequence. Contradiction.

Exercise 2 (a) Let both $\mathcal{C}$ and $\overline{\mathcal{C}}$ be Δ-elementary: $\mathcal{C} = \mathrm{Mod}(\Phi)$ and $\overline{\mathcal{C}} = \mathrm{Mod}(\Psi)$ for suitable sets $\Phi, \Psi \subseteq \mathrm{FO}(\sigma)$. Then $\Phi \cup \Psi$ is unsatisfiable as $\mathcal{C} \cap \overline{\mathcal{C}} = \emptyset$. By compactness, there is a finite unsatisfiable subset of $\Phi \cup \Psi$. Let this subset be $\Phi_0 \cup \Psi_0$, where $\Phi_0 \subseteq \Phi$ and $\Psi_0 \subseteq \Psi$ are finite. It follows that $\mathcal{C} = \mathrm{Mod}(\Phi_0)$ and $\overline{\mathcal{C}} = \mathrm{Mod}(\Psi_0)$. E.g., for $\mathcal{C}$:
$\mathcal{C} \subseteq \mathrm{Mod}(\Phi_0)$ is obvious. Suppose $\mathfrak{A} \notin \mathcal{C}$. Then $\mathfrak{A} \in \overline{\mathcal{C}}$ and hence $\mathfrak{A} \models \Psi$. Thus $\mathfrak{A} \models \Psi_0$ and as $\Phi_0 \cup \Psi_0$ is not satisfiable, it follows that $\mathfrak{A} \not\models \Phi_0$. Hence $\mathcal{C} = \mathrm{Mod}(\Phi_0)$.

(b) From the assumptions, $\Phi := \{\neg\psi \colon \psi \in \Psi\}$ is not satisfiable. By compactness, some finite subset $\Phi_0 \subseteq \Phi$ is unsatisfiable. Let $\Psi_0 := \{\psi \in \Psi \colon \neg\psi \in \Phi_0\}$. Unsatisfiability of Φ_0 directly implies that any σ-structure $\mathfrak{A}$ must satisfy at least one $\psi \in \Psi_0$.

Aside: the topological space which is here shown to be compact is the space of all complete $\mathrm{FO}(\sigma)$-theories, with basic open sets $O_\varphi := \{T \colon \varphi \in T\}$ for sentences $\varphi \in \mathrm{FO}_0(\sigma)$; we have shown that any cover by basic open sets $(O_\psi)_{\psi\in\Psi}$ admits a finite sub-cover.

Exercise 3 (a) E.g., in the standard model of arithmetic: $\mathfrak{N} \models \exists x(\underline{n} < x)$ for all $n \in \mathbb{N}$.

(b) Let $\Phi = \mathrm{Th}(\mathfrak{A}) \cup \{\varphi_i(x) \colon i \in \mathbb{N}\}$. Then Φ is finitely satisfiable (in $\mathfrak{A}$ with a suitable assignment for x), hence satisfiable by compactness.
For any model $(\mathfrak{A}', \beta) \models \Phi$ we have: $\mathfrak{A}' \equiv \mathfrak{A}$ (as $\mathfrak{A}' \models \mathrm{Th}(\mathfrak{A})$) and $\mathfrak{A}' \models \varphi_i[\beta(x)]$ for all $i \in \mathbb{N}$.

Exercise 4 Suppose first that F is continuous in 0. Then in $(\mathfrak{R}, F)$, for all n, $1 \leqslant n \in \mathbb{N}$, there is m, $1 \leqslant m \in \mathbb{N}$, such that $(\mathfrak{R}, F) \models \varphi_{n,m}$ where

$$\varphi_{n,m} = \forall x \big(\underline{m}\,|x| < 1 \rightarrow \underline{n}\,|Fx| < 1 \big).$$

Here $\underline{k}\,|z| < 1$ is used as an abbeviation for the FO formula

$$\chi_k(z) := z = 0 \vee (0 < z \wedge \underline{k}\,z < 1) \vee (z < 0 \wedge \exists z'(z + z' = 0 \wedge \underline{k}\,z' < 1)),$$

which says (in $\mathfrak{R}$) that $|z| < 1/k$.

Therefore $(\mathfrak{R}^*, F^*) \models \varphi_{n,m}$ for exactly the same pairs of $n, m \in \mathbb{N}$. If $\delta \in \mathbb{R}^*$ is in $N(0)$, then $\underline{m}\,|\delta| < 1$ is true for all $m \in \mathbb{N}$, and hence $\underline{n}|F\delta| < 1$ is true for all $n \geqslant 1$. Therefore $F^*(\delta) \in N(0)$.

If F is not continuous in 0, then there is some $n \geqslant 1$ such that

$$(\mathfrak{R}, F) \models \forall z \forall z' \big((z' < 0 \wedge 0 < z) \rightarrow (\exists x (z' < x \wedge x < z \wedge \neg\, \underline{n}\,|Fx| < 1)) \big).$$

Using the truth of this sentence in $(\mathfrak{R}^*, F^*)$, we may instantiate z and z' e.g. with δ and $-\delta$ for some positive infinitesimal δ. Then $(-\delta, \delta) \subseteq N(0)$ in $\mathfrak{R}^*$, and the above sentence says that $F^*[(-\delta, \delta)] \not\subseteq N(0)$.