Introduction to Mathematical Logic

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Solution Hints for Exercises No.6

Exercise 1

(a) Let $\mathfrak{A}, \beta \models \Phi$, $\Phi \subseteq \mathrm{FO}(\sigma)$ consisting of equality-free universal formulae. Expand $\mathfrak{T}_\sigma$ (which interprets only the constants and functions in σ) to a σ-structure $\mathfrak{T} := \mathfrak{T}_\sigma(\mathfrak{A})$ by the following stipulation (for an n-ary relation symbol $R \in \sigma$):

$$(t_1, \dots, t_n) \in R^{\mathfrak{T}} \;:\Leftrightarrow\; \mathfrak{A}, \beta \models Rt_1 \dots t_n.$$

Let β_0 be the assignment $\beta_0 \colon v_i \mapsto v_i$ in $\mathfrak{T}_\sigma$. It is then straightforward to show, by induction on formula rank, that for all equality-free universal formulae φ:

$$\mathfrak{A}, \beta \models \varphi \;\Rightarrow\; \mathfrak{T}, \beta_0 \models \varphi.$$

In particular $\mathfrak{T}, \beta_0 \models \Phi$.

(b) $\Phi \vdash \varphi \;\Rightarrow\; \mathfrak{H}, \beta^{\mathfrak{H}} \models \varphi$ is obvious for atomic and negated atomic relational formulae. The induction step for $\vee$ is based on the observation that $\Phi \vdash (\varphi_1 \vee \varphi_2)$ implies $\Phi \vdash \varphi_i$ for at least one of $i = 1, 2$, since Φ is maximally consistent (w.r.t. equality-free, universal formulae).
The induction step for $\wedge$ similarly uses that $\Phi \vdash \neg(\neg\varphi_1 \vee \neg\varphi_2)$ implies that $\Phi \vdash \varphi_i$ for $i = 1, 2$. The following derivation establishes this:

$\Gamma\ \neg(\neg\varphi_1 \vee \neg\varphi_2)$	premise
$\Gamma\ \neg\varphi_i\ \neg\varphi_i$	(Ass)
$\Gamma\ \neg\varphi_i\ (\neg\varphi_1 \vee \neg\varphi_2)$	($\vee$ S)
$\Gamma\ \neg\varphi_i\ \neg(\neg\varphi_1 \vee \neg\varphi_2)$	(Ant) on line 1
$\Gamma\ \varphi_i$	(Ctr) on previous two lines

The induction step for $\forall$: assuming the claim for all formulae of lower (quantifier) rank than $\forall x\varphi$, we show the claim for $\forall x\varphi = \neg\exists x\neg\varphi$:
$\Phi \vdash \neg\exists x\neg\varphi$ implies $\Phi \vdash \varphi\frac{t}{x}$ for all $t \in T_\sigma$, by the derived rule of $\forall$-instantiation. Therefore, $\mathfrak{H} \models \varphi\frac{t}{x}$ for all t by the induction hypothesis. Hence (by the substitution lemma) $\mathfrak{H}, \beta^{\mathfrak{H}}\frac{t^{\beta^{\mathfrak{H}}}}{x} \models \varphi$ for all t, and therefore $\mathfrak{H} \models \forall x\varphi$.
Remark: These arguments go through for universal, but not necessarily equality-free formulae in the Henkin model $\mathfrak{H}(\Phi)$ based on the quotient w.r.t. derivable term equalities.

Exercise 2

(a) (i) If Φ is inconsistent w.r.t. σ' then there are derivations $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$ in the sequent calculus over $\mathrm{FO}(\sigma')$, for some finite $\Gamma \subseteq \Phi$. Let $z \in \mathrm{Var}$ such that z does not occur anywhere in these two derivations (as the entire derivations are finite, so is the set of variable symbols that occur).
We claim that syntactic replacement of c by z throughout these derivations turns these into derivations in the sequent calculus over $\mathrm{FO}(\sigma)$. One checks

that this replacement turns each rule of $\mathcal{S}_{\sigma'}$ into a rule of $\mathcal{S}_\sigma$. That z does not occur anywhere in the old derivation implies that no application of the rule ($\exists$ A) causes problems in this transformation. So Φ would also be inconsistent w.r.t. FO(σ).

(ii) By the same argument as in (a), based on the corresponding arguments in the lecture where we used a fresh variable symbol for a witness. If $\Phi \cup \{\exists x\varphi \to \varphi\frac{c}{x}\}$ is inconsistent, then $\Phi \vdash \neg\varphi\frac{c}{x}$ and $\Phi \vdash \exists x\varphi$. Therefore, as in (a) and over σ, $\Phi_0 \vdash \exists x\varphi$ and $\Phi_0 \vdash \neg\varphi\frac{z}{x}$ for some suitable finite $\Phi_0 \subseteq \Phi$ and variable z which in particular does not occur free in Φ_0 or $\exists x\varphi$. As in the argument in the lecture, it follows that also $\Phi_0 \vdash \neg\exists x\varphi$, so Φ would have to be inconsistent.

(iii) Let $\hat{\sigma} = \sigma \cup \{c_i : i \in \mathbb{N}\}$ for new constant symbols c_i. Let $(\varphi_i)_{i\in\mathbb{N}}$ be an enumeration of FO($\hat{\sigma}$), and define a chain $\Phi = \Phi_0 \subseteq \Phi_1 \subseteq \cdots$ by induction on $i \in \mathbb{N}$ with

$$\Phi_{i+1} := \begin{cases} \Phi_i \cup \{\exists x\varphi \to \varphi\frac{c_j}{x}\} & \text{if } \varphi_i = \exists x\varphi \\ \Phi_i & \text{else} \end{cases}$$

where j is chosen as the least index j such that such that c_j does not occur in Φ_i or φ_i. Note that this is always possible as Φ_0 has no occurrences of new constants, and each Φ_i therefore only has finitely many. $\hat{\Phi} := \bigcup_i \Phi_i$ has witnesses and is consistent.

(b) Let cons$_\sigma(\Phi)$. Then, by completeness, there is a model (σ-structure with assignment) $(\mathfrak{A}, \beta) \models \Phi$. Expand $\mathfrak{A}$ to a σ'-structure $\mathfrak{A}'$ by choosing arbitrary interpretations for the symbols in $\sigma' \setminus \sigma$. By the coincidence lemma, $(\mathfrak{A}', \beta) \models \Phi$, so that sat$_{\sigma'}(\Phi)$. By correctness (soundness) we conclude that cons$_{\sigma'}(\Phi)$ (no contradiction can be derivable in the sequent claculus for σ' either).

Exercise 3

(a) Let $\mathfrak{A}$ be a σ-structure, $\varphi(\mathbf{x}, y) \in$ FO(σ). We may interpret the new function symbol $f := f_{\varphi\mathbf{x}y}$ as an n-ary function over A as follows. For $\mathbf{a} \in D := \{\mathbf{a} \in A^n : \mathfrak{A}, \mathbf{a} \models \exists y\varphi\}$, let $f(\mathbf{a}) \in \{b \in A : \mathfrak{A}, \mathbf{a}b \models \varphi\}$; for $\mathbf{a} \notin D$ let $f(\mathbf{a})$ be any element of A. [Note that the existence of such a function $f : A^n \to A$ relies on the axiom of choice.]

(b) By induction on $\psi(\mathbf{x}) \in$ FO(σ), we find $\hat{\psi}(\mathbf{x})$ such that $\mathrm{Sk}_0(\sigma) \models \forall\mathbf{x}(\psi \leftrightarrow \hat{\psi})$. Consider the $\exists$-step, for $\psi(\mathbf{x}) = \exists y\varphi(\mathbf{x}, y)$: clearly $\hat{\psi}(\mathbf{x}) := \varphi\frac{f_{\varphi\mathbf{x}y}\mathbf{x}}{y}$ is as required.

Exercise 4

(a) The set of all (graphs of) partial choice functions for $(A_i)_{i\in I}$ with the inclusion relation is closed under unions of chains, hence inductive. Any maximal partial choice function must be total, as otherwise there would be a proper extension.

(b) If $(A_i)_{i\in I}$ with non-empty A_i, by a) there exists $f : I \to \cup_{i\in I} A_i$ such that $f(i) \in A_i$ for all i. So f is actually in the Cartesian product $\bigotimes_{i\in I} A_i$.

(c) Consider the set $S := \{R : R \subseteq A \times A$ well-orders some subset $B \subseteq A\}$ with the relation $\prec$ for which $R \prec R'$ if R' is a proper end-extension of R: if R well-orders B, then R' well-orders some $B' \supsetneq B$ in such a manner that all elements of $B' \setminus B$ are larger than all elements of B in sense of R. One checks that S is closed under unions of chains, hence is inductive [convince yourself with an example that plain inclusion between well-orderings of subsets would not be good enough for this!]. Any maximal element of $(S, \prec)$ must be a well-ordering of all of A, as any missing point could just be appended to obtain a larger well-ordering.