Solution Hints Exercises No.4

Exercise 1

(a)

$$\frac{\begin{array}{ll}\Gamma & (\varphi \rightarrow \psi) \\ \Gamma & \varphi\end{array}}{\Gamma\ \psi} \quad \text{(modus ponens)}$$

$$\frac{\Gamma\ \varphi\ \neg\psi}{\Gamma\ \psi\ \neg\varphi} \quad \text{(contr 2)}$$

1	$\Gamma\ (\neg\varphi \vee \psi)$	premise
2	$\Gamma\ \varphi$	premise
3	$\Gamma\ \neg\varphi\ \varphi$	(Ant) on 2
4	$\Gamma\ \neg\varphi\ \neg\varphi$	(Ass)
5	$\Gamma\ \neg\varphi\ \psi$	(Ctr') on $3, 4$
6	$\Gamma\ \psi\ \psi$	(Ass)
7	$\Gamma\ (\neg\varphi \vee \psi)\ \psi$	($\vee$ A) on $5, 6$
8	$\Gamma\ \psi$	(chain) on $1, 7$

1	$\Gamma\ \varphi\ \neg\psi$	premise
2	$\Gamma\ \psi\ \varphi\ \neg\psi$	(Ant) on 1
3	$\Gamma\ \psi\ \varphi\ \psi$	(Ass)
4	$\Gamma\ \psi\ \varphi\ \neg\varphi$	(Ctr') on $2, 3$
5	$\Gamma\ \psi\ \neg\varphi\ \neg\varphi$	(Ass)
6	$\Gamma\ \psi\ \neg\varphi$	(CD) on $4, 5$

(b)

$$\frac{\Gamma\ \neg\neg\varphi}{\Gamma\ \varphi} \quad \text{(NN 1)}$$

$$\frac{\Gamma\ \varphi}{\Gamma\ \neg\neg\varphi} \quad \text{(NN 2)}$$

1	$\Gamma\ \neg\neg\varphi$	premise
2	$\Gamma\ \neg\varphi\ \neg\neg\varphi$	(Ant) on 1
3	$\Gamma\ \neg\varphi\ \neg\varphi$	(Ass)
4	$\Gamma\ \varphi$	(Ctr) on $2, 3$

1	$\Gamma\ \varphi$	premise
2	$\Gamma\ \neg\neg\neg\varphi\ \varphi$	(Ant) on 1
3	$\Gamma\ \neg\neg\neg\varphi\ \neg\varphi$	$(*)$
4	$\Gamma\ \neg\neg\varphi$	(Ctr) on $2, 3$

$(*)$ is the derived sequent $\Gamma\ \neg\neg\psi\ \psi$, for $\psi = \neg\varphi$. This sequent (a variant of the first double negation rule) is easily derived from $\Gamma\neg\psi\neg\psi$ through an application of the first contrapositive rule.

Exercise 2

(a) For $\wedge$, the following are derivable (and semantically adequate):

$$\frac{\Gamma\varphi_1\varphi_2\ \ \psi}{\Gamma(\varphi_1 \wedge \varphi_2)\ \psi} \quad (\wedge\text{A})$$

$$\frac{\begin{array}{ll}\Gamma & \varphi_1 \\ \Gamma & \varphi_2\end{array}}{\Gamma\ (\varphi_1 \wedge \varphi_2)} \quad (\wedge\text{S})$$

(b) For $\rightarrow$ the following are derivable (and semantically adequate):

$$\frac{\begin{array}{ll}\Gamma\psi & \chi \\ \Gamma\neg\varphi & \chi\end{array}}{\Gamma(\varphi \rightarrow \psi)\ \chi} \quad (\rightarrow\text{A})$$

$$\frac{\Gamma\varphi\ \ \psi}{\Gamma\ (\varphi \rightarrow \psi)} \quad (\rightarrow\text{S})$$

Exercise 3

$$\frac{\Gamma\ \neg\exists x\neg\varphi}{\Gamma\ \varphi\frac{t}{x}}$$

1	$\Gamma\ \neg\exists x\neg\varphi$	premise
2	$\Gamma\ \neg\varphi\frac{t}{x}\ \neg\varphi\frac{t}{x}$	(Ass)
3	$\Gamma\ \neg\varphi\frac{t}{x}\ \exists x\neg\varphi$	($\exists$ S)
4	$\Gamma\ \neg\varphi\frac{t}{x}\ \neg\exists x\neg\varphi$	(Ant) on 1
5	$\Gamma\ \varphi\frac{t}{x}$	(Ctr) on $3, 4$

Exercise 4 ($*$) refers to use of the derived rule from the previous exercise; let

$$\gamma = \neg\exists x\neg\exists y\ x{=}fy$$

be the antecedent of the desired sequent.

1	$c{=}fy \quad\quad c{=}fy$	(Ass)
2	$c{=}fy \quad y{=}fz \quad c{=}ffz$	(Sub) on 1
3	$c{=}fy \quad y{=}fz \quad \exists z\, c{=}ffz$	($\exists$S) on 2
4	$c{=}fy \quad \exists z\, y{=}fz \quad \exists z\, c{=}ffz$	($\exists$A) on 3
		z not free in $c{=}fy, \exists z\, y{=}fz, \exists z\, c{=}ffz$
5	$\gamma \quad \gamma$	(Ass)
6	$\gamma \quad \exists z\, y{=}fz$	($*$) on 5 with $t = y$
		renaming bound y to z!
7	$\gamma \quad c{=}fy \quad \exists z\, y{=}fz$	(Ant) on 6
8	$\gamma \quad c{=}fy \quad \exists z\, y{=}fz \quad \exists z\, c{=}ffz$	(Ant) on 4
9	$\gamma \quad c{=}fy \quad \exists z\, c{=}ffz$	(chain) on $7, 8$
10	$\gamma \quad \exists y\, c{=}fy \quad \exists z\, c{=}ffz$	($\exists$ A) on 9
		y not free in $\gamma, \exists x\, c{=}fx, \exists z\, c{=}ffz$
11	$\gamma \quad \exists y\, c{=}fy$	($*$) on 5 with $t = c$
12	$\gamma \quad \exists z\, c{=}ffz$	(chain) on $10, 11$