Introduction to Mathematical Logic

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Solution Hints for Exercises No.3

Exercise 1 (a) (i) Let $\sigma := \{P_f, P_v, +, \cdot, 0_f, 1_f, \oplus, \otimes, 0_v\}$ where P_f and P_v are unary relation symbols (intended for the number domain and vector domain), $+$, $\cdot$, $\oplus$ and $\otimes$ are binary function symbols, and 0_f, 1_f and 0_v are constant symbols. The closure conditions on the number domain are:

- $\forall x \forall y((P_f x \wedge P_f y) \to P_f(x \cdot y))$
- $\forall x \forall y((P_f x \wedge P_f y) \to P_f(x + y))$
- $P_f 0_f$ and $P_f 1_f$

Besides, one uses 'relativisations' of the field axioms to the number domain, in which existential quantification $\exists x...$ is replaced by $\exists x(P_f x \wedge ...)$ and universal quantification $\forall x...$ by $\forall x(P_f x \to ...)$.

[NB: this process can be defined systematically by induction; question: what is the semantic criterion for the relationship between φ and its relativisation φ' to some unary predicate P that does not occur in φ ?]

The vector space axioms are similarly obtained by suitable relativisations of quantifications to P_f and to P_v, as appropriate.

[NB: we do not care how the operations behave on arguments of inappropriate sorts!]

(ii) Let $\sigma := \{\oplus, 0_v, f_{\lambda_1}, \ldots, f_{\lambda_p}\}$, where $\lambda_1, \ldots, \lambda_p$ are the elements of the field $\mathbb{F}_p$. As above, $\oplus$ is interpreted as vector addition and 0_v as its neutral element. Each f_λ, a unary function symbol, is interpreted as multiplication of a vector by the field element λ. Then a typical vector space axiom like 'associativity' for scalar multiplication looks like this:

$\forall x f_\lambda(f_\mu(x)) = f_{\lambda\mu}(x)$ [one for every pair of field elements λ, μ]

or, a distributivity axiom, like this:

$\forall x \forall y(f_\lambda(x \oplus y) = f_\lambda(x) \oplus f_\lambda(y))$ [one for every field element λ]

(b) (i) Let the notation $x \leqslant y$ stand for the formula $x = y \vee x < y$ and let $\delta(x, y, z) := 0 \leqslant z \wedge ((x \leqslant y \wedge y \leqslant x + z) \vee (y \leqslant x \wedge x \leqslant y + z))$, which says that $|x - y| \leqslant z$. Then continuity in 0 is expressed by
$\forall z(0 < z \to \exists z'(0 < z' \wedge \forall x(\delta(x, 0, z') \to \delta(fx, f0, z))))$

Exercise 2 The inductive definition of a negation normal form map may be given as follows, based on our full FO syntax with $\to$, $\leftrightarrow$ for completeness. We simultaneously define the values for the two functions $f(\varphi) = \mathrm{nnf}(\varphi)$ and $g(\varphi) = \mathrm{nnf}(\neg\varphi)$.

(F1), (F2): $f(\varphi) := \varphi$, $g(\varphi) := \neg\varphi$ for atomic φ.

(F3): $f(\neg\varphi) := g(\varphi)$
$g(\neg\varphi) := f(\varphi)$.

(F4) $\vee$ and $\wedge$: $\begin{array}{l} f(\varphi_1 * \varphi_2) := f(\varphi_1) * f(\varphi_2) \\ g(\varphi_1 * \varphi_2) := g(\varphi_1) \bar{*} g(\varphi_2) \end{array}$ for $* = \vee, \wedge$ and $\bar{\vee} := \wedge, \bar{\wedge} := \vee$.

(F4) $\rightarrow$: $\begin{array}{l} f(\varphi_1 \rightarrow \varphi_2) := g(\varphi_1) \vee f(\varphi_2) \\ g(\varphi_1 \rightarrow \varphi_2) := f(\varphi_1) \wedge g(\varphi_2). \end{array}$

(F4) $\leftrightarrow$: $f(\varphi_1 \leftrightarrow \varphi_2) = f(\varphi_1 \leftrightarrow \varphi_2) := (f(\varphi_1) \wedge f(\varphi_2)) \vee (g(\varphi_1) \wedge g(\varphi_2))$.

(F5) $\begin{array}{l} f(Qx\varphi) := Qx\, f(\varphi) \\ g(Qx\varphi) := \bar{Q}x\, g(\varphi) \end{array}$ for $Q = \forall, \exists$ and $\bar{\forall} := \exists, \bar{\exists} := \forall$.

The inductive proof of the adequacy of these stipulations is then straightforward.

Exercise 3 The claim is shown by syntactic induction on the formula part φ of the game position. We call a position in which V (R) has a winning strategy a *winning position for V (R)*. Along with the proof of the claim as stated we may establish that a position (φ, β) is winning for R iff $(\mathfrak{A}, \beta) \not\models \varphi$ iff it is not winning for V.

If φ is atomic or negated atomic, then the game has already terminated, and R and V have lost or won (have a trivial winning strategy) in accordance with the claim.

Consider a game position (φ, β) with $\varphi = (\varphi_1 \vee \varphi_2)$. Then it is V's move. Thus (φ, β) is winning for V iff at least one of the target positions she can move to is winning for her, i.e., by the inductive hypothesis, iff $(\mathfrak{A}, \beta) \models \varphi_i$ for at least one of $i = 1, 2$, hnece iff $(\mathfrak{A}, \beta) \models \varphi$.

The dual cases ($\vee$-position for R, or $\wedge$-position for V) are treated analogously.

Consider a game position (φ, β) with $\varphi = \exists x \varphi$. Then it is V's move. Thus (φ, β) is winning for V iff at least one of the target positions she can move to is winning for her, i.e., iff for at least one $a \in A$, $(\varphi, \beta\frac{a}{x})$ is winning for her, iff, by the inductive hypothesis, $(\mathfrak{A}, \beta\frac{a}{x}) \models \varphi$ for at least one $a \in A$, and hence iff $(\mathfrak{A}, \beta) \models \varphi$.

Again, the dual cases ($\exists$-position for R, or $\forall$-position for V) are treated analogously.

If formulae are not assumed to be in nnf: in positions $(\neg\varphi, \beta)$, let V and R swap roles and then proceed from position (φ, β).

Formally, one may add a tag $\wp$ to each position that tells which of the two players, I or II say, currently acts as the verifier. Then the rules for moves as given just preserve $\wp$, while we have a forced move from $(\neg\varphi, \beta, \mathrm{I})$ to $(\varphi, \beta, \mathrm{II})$ and from $(\neg\varphi, \beta, \mathrm{II})$ to $(\varphi, \beta, \mathrm{I})$. Terminating positions are now of the form $(\varphi, \beta, \wp)$ for atomic φ, and $\wp$ wins iff $\mathfrak{A}, \beta \models \varphi$.

Then player I has a winning strategy in the game starting in $(\varphi, \beta, \mathrm{I})$ on $\mathfrak{A}$ iff $\mathfrak{A}, \beta \models \varphi$.