Introduction to Mathematical Logic Martin Otto
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Solution Hints for Exercises No.2

Exercise 1 (a) Base case, t is a constant c or a variable x, so the only proper prefix of t is the empty word, which is not a term in T_σ. Also note that any term which is neither a constant nor a variable starts with a function symbol, and so does every non-empty prefix of it. Inductive case, assume that $t = ft_1 \dots t_n$ where $t_1, \dots, t_n$ satisfy the property to be proved. Let t' be a term s.t. $ft_1 \dots t_n \sqsubseteq t'$ or $t' \sqsubseteq ft_1 \dots t_n$, where $u \sqsubseteq u'$ means that u is a prefix of u', and note that t' must be of the form $fu_1 \dots u_n$ for some terms $u_1, \dots, u_n$. Let us prove by induction on $k \in \mathbb{N}$ that if $k \leqslant n$ then for all $i \leqslant k$ we have $u_i = t_i$. Case $k = 0$, trivial. If the property holds for k, then either $n \leqslant k$, in which case it is trivial, or we obtain $u_{k+1} \dots u_n = t_{k+1} \dots t_n$. In the latter case, $u_{k+1} \sqsubseteq t_{k+1}$ or $t_{k+1} \sqsubseteq u_{k+1}$ and we can apply the induction hypothesis.

(b) Let $U_\sigma(V)$ be the set defined as follows: the constants of σ are in $U_\sigma(V)$, the variables in V are in $U_\sigma(V)$, and for all function symbols f of σ and all $u_1, \dots, u_n$ in $U_\sigma(V)$ where n is the arity of f, the word $fu_1 \dots u_n$ is also in $U_\sigma(V)$. To answer the question it suffices to prove by (syntactic) induction that for all terms t in T_σ, the term t is in $T_\sigma(V)$ iff it is in $U_\sigma(V)$, which requires to refer to the inductive definition of the function var returning the set of variables involved in a term.

Exercise 2 (a) Verify the closure conditions for constant and function symbols.

(b) For instance show by induction on $t \in T_\sigma(\emptyset)$ (or for $t \in T_\sigma$ with $\mathrm{var}(t) = \emptyset$) that for any homomorphism $h\colon \mathfrak{T}_\sigma(\emptyset) \xrightarrow{\text{hom}} \mathfrak{A}$ and any assignment $\beta\colon \mathrm{Var} \to A$, h must coincide with the interpretation function $\mathfrak{I}^{\mathfrak{A}}_\beta$ on $T_\sigma(\emptyset)$.

(c) Similar to part (b), but for any $\beta\colon \mathrm{Var} \to A$ that extends β_0.

Exercise 3 Avoiding some (semantically redundant) parentheses, we have that for instance $\mathcal{C}_i = \mathrm{Mod}(\varphi_i)$ for

$$\varphi_1 := \forall v_0 \forall v_1 \forall v_2 \left(\begin{array}{l} \neg Rv_0v_0 \\ \wedge \big(Rv_0v_1 \vee Rv_1v_0 \vee v_0 = v_1\big) \\ \wedge \big((Rv_0v_1 \wedge Rv_1v_2) \to Rv_0v_2\big) \end{array} \right) \wedge \forall v_0 \exists v_1 Rv_0v_1.$$

[this is for linear orderings in the sense of <]

$$\begin{array}{rl} \varphi_2 := & \forall v_0 \exists v_1 Rv_0v_1 \\ & \wedge \forall v_0 \forall v_1 \forall v_2 \big((Rv_0v_1 \wedge Rv_0v_2) \to v_1 = v_2\big) \\ & \wedge \forall v_0 \forall v_1 \forall v_2 \big((Rv_0v_2 \wedge Rv_1v_2) \to v_0 = v_1\big) \\ & \wedge \exists v_1 \forall v_0 \neg Rv_0v_1. \end{array}$$

$$\varphi_3 := \forall v_0 \forall v_1 \forall v_2 \Big(Rv_0v_0 \wedge \big(Rv_0v_1 \leftrightarrow Rv_1v_0\big) \wedge \big((Rv_0v_1 \wedge Rv_1v_2) \to Rv_0v_2\big) \Big) \wedge \exists v_0 \exists v_1 \neg Rv_0v_1.$$

$$\varphi_4 := \exists v_1 \exists v_2 \exists v_3 \Big(\bigwedge_{1 \leqslant i < j \leqslant 3} \neg v_i = v_j \wedge \bigwedge_{(i,j) \in R^{\mathfrak{A}}} Rv_iv_j \wedge \bigwedge_{(i,j) \notin R^{\mathfrak{A}}} \neg Rv_iv_j \wedge \forall v_0 \big(\bigvee_{1 \leqslant i \leqslant 3} v_0 = v_i \big) \Big).$$

Exercise 4 Just as in the proof of the isomorphism lemma we find in preparation for the main proof that for all $t \in T_S$

$$\mathfrak{I}^{\mathfrak{B}}_{h\circ\beta}(t) = h\big(\mathfrak{I}^{\mathfrak{A}}_{\beta}(t)\big).$$

(Injectivity, surjectivity or strictness of the homomorphism are not required here yet.)

The essential steps in the induction on φ for the main claim:

(F1) $\varphi = t = t'$: $(\mathfrak{A}, \beta) \models t = t' \;\Leftrightarrow\; \mathfrak{I}^{\mathfrak{A}}_{\beta}(t) = \mathfrak{I}^{\mathfrak{A}}_{\beta}(t')$

$$\begin{aligned} &\Rightarrow\; h\big(\mathfrak{I}^{\mathfrak{A}}_{\beta}(t)\big) = h\big(\mathfrak{I}^{\mathfrak{A}}_{\beta}(t')\big)\\ &\Leftrightarrow\; \mathfrak{I}^{\mathfrak{B}}_{h\circ\beta}(t) = \mathfrak{I}^{\mathfrak{B}}_{h\circ\beta}(t')\\ &\Leftrightarrow\; (\mathfrak{B}, h\circ\beta) \models t = t'. \end{aligned}$$

(F2) analogous, using that h preserves membership of tuples in R from $\mathfrak{A}$ to $\mathfrak{B}$.

(F4) for $\vee$ and $\wedge$: straightforward!

(F5) for $\varphi = \exists x\psi$.

If $(\mathfrak{A}, \beta) \models \varphi$, then $(\mathfrak{A}, \beta\frac{a}{x}) \models \psi$ for some $a \in A$. It follows (by inductive hypothesis) that $(\mathfrak{B}, h\circ(\beta\frac{a}{x})) \models \psi$, which implies that $(\mathfrak{B}, (h\circ\beta)\frac{h(a)}{x})) \models \psi$, hence $(\mathfrak{B}, h\circ\beta) \models \varphi$.

(F5) for $\varphi = \forall x\psi$.

If $(\mathfrak{A}, \beta) \models \varphi$, then $(\mathfrak{A}, \beta\frac{a}{x}) \models \psi$ for all $a \in A$. It follows that for all $a \in A$: $(\mathfrak{B}, h\circ(\beta\frac{a}{x})) \models \psi$, which implies that $(\mathfrak{B}, (h\circ\beta)\frac{h(a)}{x})) \models \psi$ for all $a \in A$. As h is surjective (the only place where this is needed), $(\mathfrak{B}, (h\circ\beta)\frac{b}{x})) \models \psi$ for all $b \in B$ and therefore $(\mathfrak{B}, h\circ\beta) \models \varphi$.

Examples precluding stronger versions:

Let $\sigma = \{f, c\}$, f a unary function symbol. Then $\forall x fx = c$ is true in the one-element σ-structure $\mathfrak{A} = (\{0\}, \mathrm{id}, 0)$, but not in $\mathfrak{B} = (\{0,1\}, \mathrm{id}, 0)$, despite the obvious homomorphism (which fails to be surjective).

W.r.t. the requirement of positivity consider the formula $\neg v_0 = v_1$ (over $\sigma = \emptyset$), and the unique (surjective) homomorphism from $\mathfrak{A} = (\{0,1\})$ to $\mathfrak{B} = (\{0\})$.

Exercise 5 For instance, for $\forall x\forall y\varphi \equiv \forall y\forall x\varphi$. If $x = y$ (same variable symbol) the assertion is trivial, so we assume that x and y are distinct variable symbols. This implies that for any assignment $\beta\colon \mathrm{Var} \to A$ and all $a, a' \in A$, $(\beta\frac{a}{x})\frac{a'}{y} = (\beta\frac{a'}{y})\frac{a}{x}$. Accordimg to the definition of the semantics of universal quantification (used twice), $(\mathfrak{A}, \beta) \models \forall x\forall y\varphi$ iff for all $a, a' \in A$ $(\mathfrak{A}, (\beta\frac{a'}{y})\frac{a}{x}) \models \varphi$. By the above this is the case iff for all $a', a \in A$ $(\mathfrak{A}, (\beta\frac{a}{x})\frac{a'}{y}) \models \varphi$ iff $(\mathfrak{A}, \beta) \models \forall y\forall x\varphi$.