Introduction to Mathematical Logic

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Solution Hints for Exercises No.13

Exercise 1

(a) Example 1: for countably many unary relations $(P_i)_{i\in\omega}$, let $\mathfrak{A} = (\mathrm{P}(\omega), (P_i)^{\mathfrak{A}}_{i\in\omega})$ where $(P_i)^{\mathfrak{A}} = \{s \subseteq \omega : i \in s\}$. Note that for every element $s \in \mathrm{P}(\omega)$, $s = \{i \in \omega : s \in P_i^{\mathfrak{A}}\}$. Let $\mathfrak{B} \equiv \mathfrak{A}$ be any countable model of $\mathrm{Th}(\mathfrak{A})$ (recall the downward Löwenheim–Skolem theorem). Then there are subsets $s \in \mathrm{P}(\omega)$ for which there is no $b \in B$ such that $\mathfrak{A}, s \equiv_0 \mathfrak{B}, b$. This implies that not even $\mathfrak{A} \simeq_1 \mathfrak{B}$.
Example 2: Consider the set $\mathbb{Q}$ of rationals with unary relations $I_n := (-1/n, 1/n)$ for $n \geqslant 1$ (nested intervals such that $I_n \setminus I_{n+1}$ is infinite). Call this structure $\mathfrak{A}$, and let $\mathfrak{A}^\circ$ be the restriction of $\mathfrak{A}$ to $\mathbb{Q} \setminus \{0\}$. Then $\mathfrak{A} \not\simeq_1 \mathfrak{A}^\circ$ as there is no partial isomorphism with $0 \in A$ (this is the unique element that is a member of all I_n in $\mathfrak{A}$, while the intersection of the I_n is empty in $\mathfrak{A}^\circ$). The reducts of $\mathfrak{A}$ and $\mathfrak{A}^\circ$ to any finite subset of the I_n are not only elementarily equivalent, but even isomorphic! It follows that $\mathfrak{A} \equiv \mathfrak{A}^\circ$.

(b) Let $\mathfrak{A} \simeq_{\mathrm{fin}} \mathfrak{B}$ and, for instance, $\mathfrak{A}$ finite. Let $A = \{a_1, \dots, a_n\}$ and use the (forth)-property in some $(I_k)_{k\in\mathbb{N}} : \mathfrak{A} \simeq_{\mathrm{fin}} \mathfrak{B}$ to obtain $b_1, \dots, b_n$ such that $\mathfrak{A}, a_1, \dots, a_n \simeq_1 \mathfrak{B}, b_1, \dots, b_n$. It follows that $B = \{b_1, \dots, b_n\}$ and $p : (a_i \mapsto b_i)_{1\leqslant i\leqslant n}$ is an isomorphism. In fact, $(I_k)_{k\leqslant n+1} : \mathfrak{A} \simeq_{n+1} \mathfrak{B}$ suffices for the argument to go through.

Exercise 2

We show that $p : \bar{a} \mapsto \bar{b}$ in $\dot{I}_{\ell+1}$, has forth extensions in $\dot{I}_\ell$ for every $a \in A$. According to the definition of $\dot{I}_{\ell+1}$, the a_i and b_i respect $d^{\ell+1}$. Let $a \in A$, w.l.o.g. $a \in (a_i, a_{i+1})^{\mathfrak{A}}$ (the open interval w.r.t. $<^{\mathfrak{A}}$) for some i. We need to find some $b \in (b_i, b_{i+1})^{\mathfrak{B}}$ such that still $d^\ell(b_i, b) = d^\ell(a_i, a)$ and $d^\ell(b, b_{i+1}) = d^\ell(a, a_{i+1})$.

If $d^{\ell+1}(b_i, b_{i+1}) = d^{\ell+1}(a_i, a_{i+1})$ is finite, then an 'isomorphic copy' of the position of a into $(b_i, b_{i+1})^{\mathfrak{B}}$ will do.

Otherwise, both intervals have lengths at least $2^{\ell+1}$ and at most one of the distances $d^\ell(a_i, a)$ or $d^\ell(a, a_{i+1})$ can be finite. If neither is finite, find b such that neither $d^\ell(b_i, b)$ or $d^\ell(b, b_{i+1})$ is finite (note that $d^{\ell+1}(b_i, b_{i+1}) \geqslant 2^{\ell+1}$); else copy the finite ℓ-distance exactly, and the other one will automatically be infinite.

(a) Assume $\varphi \in \mathrm{FO}_0(<)$ is satisfied precisely by those finite linear ordering whose length is even. For $n := \mathrm{qr}(\varphi)$ consider the finite discrete linear orderings $\mathfrak{A}$ of length $2^n + 1$ and $\mathfrak{B}$ of length $2^n + 2$, respectively. For these, $d^n(a_{\min}, a_{\max}) = d^n(b_{\min}, b_{\max}) = \infty$. Therefore $\mathfrak{A}, a_{\min}, a_{\max} \simeq_n \mathfrak{B}, b_{\min}, b_{\max}$, hence $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$, but only $\mathfrak{B}$ has even length. (The last assertion is even true for $\mathfrak{A} \restriction (a_{\min}, a_{\max})^{\mathfrak{A}}$ vs. $\mathfrak{B} \restriction (b_{\min}, b_{\max})^{\mathfrak{B}}$, which have lengths $2^n - 1$ and 2^n.)

(b) Let $\mathfrak{A} = (\mathbb{N}, <)$ and $\mathfrak{B} =$ "$(\mathbb{N}, <) + (\mathbb{Z}, <)$". Note that both orderings are discrete, both with first and without last element. In order to show that $\mathfrak{A} \simeq_m \mathfrak{B}$, though, it suffices to show the following. $\mathfrak{A}$ and $\mathfrak{B}$ can each be presented as the ordered sum of an initial segment and a final segment, $\mathfrak{A} = \mathfrak{A}^{(0)} + \mathfrak{A}^{(1)}$ and $\mathfrak{B} = \mathfrak{B}^{(0)} + \mathfrak{B}^{(1)}$ in such a manner that $\mathfrak{A}^{(i)} \simeq_m \mathfrak{B}^{(i)} i$ for $i = 0, 1$: back-and-forth systems $(I_k^{(0)})_{k\leqslant m}$ and

$(I_k^{(1)})_{k\leqslant m}$ for the individual $\simeq_m$-relationships can be combined to form a back-and-forth system $(I_k)_{k\leqslant m}$ for $\mathfrak{A} \simeq_m \mathfrak{B}$, where I_k consists of all disjoint combinations of $p^{(0)} \in I_k^{(0)}$ and $p^{(1)} \in I_k^{(1)}$.
For a given m, let $\mathfrak{A}^{(0)}$ be an initial segment of $\mathfrak{A}$ of length greater than 2^m, $\mathfrak{A}^{(1)}$ the rest of $\mathfrak{A}$ (which is isomorphic to $(\mathbb{N}, <)$); similarly let $\mathfrak{B}^{(1)}$ a final segment of $\mathfrak{B}$ that is isomorphic to $(\mathbb{N}, <)$, and $\mathfrak{B}^{(0)}$ the remaining initial segment (which is isomorphic to $(\mathbb{N}, <)$ with a reverse copy of $(\mathbb{N}, <)$ appended on the right). Then the final segments $\mathfrak{A}^{(1)}$ and $\mathfrak{B}^{(1)}$ are isomorphic (hence trivially $\simeq_m$ equivalent), and the initial segments $\mathfrak{A}^{(0)}$ and $\mathfrak{B}^{(0)}$ are $\simeq_m$ equivalent (as discrete linear orderings with first and last elements) by the main part of this exercise (note that both are "infinitely long" in the sense of d^m).

Exercise 3 Clearly the set I of all finite monotone maps between two dense linear orderings without endpoints is a back-and-forth system. For countable $\mathfrak{A} = (A, <^{\mathfrak{A}})$ and $\mathfrak{A} = (B, <^{\mathfrak{B}})$, enumerate A as $(a_i)_{i\in\mathbb{N}}$ and B as $(b_i)_{i\in\mathbb{N}}$, and use the fact that $I\colon \mathfrak{A} \simeq_{\text{part}} \mathfrak{B}$ to find an increasing sequence of finite mononotone maps $p \in \text{Part}(\mathfrak{A}, \mathfrak{B})$ $(p_n)_{n\in\mathbb{N}}$ such that $a_n \in \text{dom}(p_n)$ and $b_n \in \text{image}(p_n)$. Then $f := \bigcup_n p_n$ is well-defined, monotone and a bijection; hence an isomorphism.