Introduction to Mathematical Logic | Martin Otto
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Solution Hints for Exercises No.12

Exercise 1 Let us reason semantically, for $\models$ instead of $\vdash$ (completeness thm). [You may think of working in an arbitray but fixed model of Q.]

(a) We show the claim by induction on $n \in \mathbb{N}$. Base case. For $m > 0$, $Q \models \neg\, \underline{0} = \underline{m}$. Note that $\underline{0} = 0$ and that $m = \dot{m} + 1$ for $\dot{m} := m - 1$ implies that $\underline{m} = \underline{\dot{m}} + 1$. Hence the claim follows from Q1.
Induction step. For $m > n+1$, $Q \models \neg\, \underline{n+1} = \underline{m}$: as $\underline{m} = \underline{\dot{m}}+1$ and $\underline{n+1} = \underline{n}+1$, by Q2 the claim follows from the inductive hypothesis that $Q \models \neg\, \underline{n} = \underline{\dot{m}}$, as $\dot{m} = m - 1 > n$.

(b) We want to use $\varphi_+(x, y, z) := x + y = z$ to represent addition. We merely need to show that $Q \models \underline{n} + \underline{m} = \underline{n+m}$ for all $n, m \in \mathbb{N}$; uniqueness of the corresponding z even holds in the strong sense that (by functionality of $+$, irrespective of Q) $\vdash \forall x \forall y \exists^{=1} z\; x + y = z$. $Q \models \underline{n} + \underline{m} = \underline{n+m}$ is easily established by induction on m on the basis of Q4 and Q5.

(c) Suppose that for all $n, m \in \mathbb{N}$: $Q \models \varphi_f(\underline{n}, \underline{f(n)}) \wedge \forall z(\varphi_f(\underline{n}, z) \rightarrow z = \underline{f(n)})$, and similarly $Q \models \varphi_g(\underline{m}, \underline{g(m)}) \wedge \forall z(\varphi_g(\underline{m}, z) \rightarrow z = \underline{g(m)})$.
Let $\varphi_h(x, z) := \exists y(\varphi_g(x, y) \wedge \varphi_f(y, z))$
Then, for all m, putting $n := g(m)$, we have $Q \models \forall y\big(\varphi_g(\underline{m}, y) \rightarrow y = \underline{n}\big)$ and therefore also $Q \models \forall y \forall z\big((\varphi_g(\underline{m}, y) \wedge \varphi_f(y, z)) \rightarrow (y = \underline{n} \wedge z = \underline{f(n)})\big)$. Therefore $Q \models \forall z(\varphi_h(\underline{m}, z) \rightarrow z = \underline{h(m)})$. That $Q \models \varphi_h(\underline{m}, \underline{h(m)})$ is obvious.

Exercise 2 (a) Let us think of working in a given model of Q and show by induction on n that $\forall x(\varphi_{\leqslant}(x, \underline{n}) \rightarrow (x = \underline{0} \vee x = \underline{1} \vee \ldots \vee x = \underline{n}))$ is satisfied.
$n = 0$: suppose $\varphi_{\leqslant}(x, \underline{0}) \wedge \neg\, x = 0$ is true for some assignment a to x. As $a \neq 0$, by Q3, $a = \dot{a} + 1$ for some $\dot{a}$. Then $b + a = 0$ implies $(b + \dot{a}) + 1 = 0$ by Q5, thus contradicting Q1.
Induction step from n to $n + 1$. Suppose $\varphi_{\leqslant}(x, \underline{n+1})$ for some assignment a to x. Then there is some b such that $b + a = \underline{n+1} = \underline{n} + 1$. If $a \neq 0$, then again $a = \dot{a} + 1$ for suitable $\dot{a}$ by Q3, and from Q5 and Q2 and we obtain $b + \dot{a} = \underline{n}$. By induction hypothesis therefore $\dot{a} \in \{\underline{0}, \ldots, \underline{n}\}$, whence the desired claim follows for $a = \dot{a} + 1$.

(b) Suppose that $m \leqslant n$. Let $k := n - m$. Then $Q \vdash \underline{k} + \underline{m} = \underline{n}$ (see previous exercise), and hence $Q \vdash \exists z(z + \underline{m} = \underline{n})$, i.e. $Q \vdash \varphi_{\leqslant}(\underline{m}, \underline{n})$.
Conversely, if $n < m$, then $Q \vdash \forall z(\varphi_{\leqslant}(z, \underline{n}) \rightarrow (z = \underline{0} \vee z = \underline{1} \vee \ldots \vee z = \underline{n}))$ and $Q \vdash \neg\, \underline{m} = \underline{\ell}$ for $0 \leqslant \ell \leqslant n$ imply that $Q \vdash \neg\varphi_{\leqslant}(\underline{m}, \underline{n})$.

(c) The map $n \mapsto \underline{n}^{\mathfrak{A}}$ from $\mathbb{N}$ to B is the required isomorphism. It is injective by Exercise 1(a), it is surjective by definition, 0 and 1 are mapped to $\underline{0}^{\mathfrak{A}}$ and $\underline{1}^{\mathfrak{A}}$ by definition, $\underline{(n + m)}^{\mathfrak{A}} = \underline{n}^{\mathfrak{A}} + \underline{m}^{\mathfrak{A}}$ by Exercise 1(b), which, together with Axioms Q6/7 yields $\underline{(n \cdot m)}^{\mathfrak{A}} = \underline{n}^{\mathfrak{A}} \cdot \underline{m}^{\mathfrak{A}}$.

(d) One can construct a model of Q whose universe consists of the disjoint union of the sets $\mathbb{N}$ and $\mathbb{Z}$. We indicate elements as $n_{\mathbb{N}}$ or $d_{\mathbb{Z}}$ to say which part of this

universe they come from. We interpret 0 and 1 as $0_\mathbb{N}$ and $1_\mathbb{N}$, respectively. The successor operation $a \mapsto a+1$ (which is part of the interpretation of $+$) is chosen to be the natural one on both parts: $n_\mathbb{N} + 1 := (n+1)_\mathbb{N}$ and $d_\mathbb{Z} + 1 := (d+1)_\mathbb{Z}$. Further interpret $+$ as usual whenever the second argument is from the $\mathbb{N}$-part: $n_\mathbb{N} + m_\mathbb{N} := (n+m)_\mathbb{N}$ and $d_\mathbb{Z} + m_\mathbb{N} := (d+m)_\mathbb{Z}$. For second argument from the $\mathbb{Z}$-part, however, we put $n_\mathbb{N} + d_\mathbb{Z} := d_\mathbb{Z}$ and $e_\mathbb{Z} + d_\mathbb{Z} := d_\mathbb{Z}$. One checks that this stipulation does respect Q4 and Q5, and multiplication can be defined to satisfy Q6 and Q7 as well (how exactly?). In this model, the relation defined by $\varphi_\leqslant$ does not link any $d_\mathbb{Z} \neq d'_\mathbb{Z}$: $x + d_\mathbb{Z} = d_\mathbb{Z} \neq d'_\mathbb{Z}$ and $x + d'_\mathbb{Z} = d'_\mathbb{Z} \neq d_\mathbb{Z}$ for all x.
Also note that, if A and B are defined as above, then for every $a \in A \backslash B$ we have $\varphi_\leqslant(\underline{n}, a)$. This can be proved by induction on n.

Exercise 3 (Löb's Theorem)
Assume $\Phi \vdash \mathrm{prov}_\Phi(\underline{\ulcorner \eta \urcorner}) \to \eta$ and let φ be a fixpoint

$$\Phi \vdash \varphi \leftrightarrow \big(\mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \to \eta\big).$$

From the latter derivability get, by (L1) and (L2), that

$$\Phi \vdash \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \to \big(\mathrm{prov}_\Phi(\underline{\ulcorner \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \to \eta \urcorner})\big).$$

We now combine this with the following instance of (L3):

$$\Phi \vdash \mathrm{prov}_\Phi(\underline{\ulcorner \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \to \eta \urcorner}) \to \Big(\mathrm{prov}_\Phi(\underline{\ulcorner \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \urcorner}) \to \mathrm{prov}_\Phi(\underline{\ulcorner \eta \urcorner})\Big),$$

to obtain (by propositional rules in $\mathcal{S}$) that

$$\Phi \vdash \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \to \big(\mathrm{prov}_\Phi(\underline{\ulcorner \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \urcorner}) \to \mathrm{prov}_\Phi(\underline{\ulcorner \eta \urcorner})\big).$$

Using (L3) for φ:

$$\Phi \vdash \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \to \mathrm{prov}_\Phi(\underline{\ulcorner \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \urcorner}),$$

we find (by modus ponens in $\mathcal{S}$):

$$\Phi \vdash \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \to \mathrm{prov}_\Phi(\underline{\ulcorner \eta \urcorner}).$$

By the assumption on η, i.e., $\Phi \vdash \mathrm{prov}_\Phi(\underline{\ulcorner \eta \urcorner}) \to \eta$, this further gives (by modus ponens in $\mathcal{S}$)

$$\Phi \vdash \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \to \eta. \qquad (*)$$

As $\Phi \vdash \varphi \leftrightarrow \big(\mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \to \eta\big)$, $(*)$ implies that $\Phi \vdash \varphi$ and, since by (L1) now also $\Phi \vdash \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner})$, we also get from $(*)$ that $\Phi \vdash \eta$.