Introduction to Mathematical Logic

Martin Otto
Stephane Le Roux
Winter 2012/13

Solution Hints for Exercises No.11

Exercise 1 $D(k,\ell) = 1/2\big((k+\ell)^2 + 3k + \ell\big)$:

One checks that for the grid points $(0,\ell)$ on the y-axis, one needs to have $D(0,\ell) = \sum_{i=1}^{\ell} i = (\ell+1)\ell/2 = 1/2(\ell^2+\ell)$. Grid point (k,ℓ) is the k-th successor along the diagonal through $(0,k+\ell)$, hence $D(k,\ell) = 1/2((k+\ell)^2 + (k+\ell)) + k$.

A formula for representing D therefore is

$$\varphi(x,y,z) := \underline{2}z = (x+y)(x+y) + \underline{3}x + y.$$

For p_1 (resp. p_2) define φ_1 (resp. φ_2) as follows.

$$\varphi_1(x,z) := \exists y \varphi(x,y,z) \qquad \varphi_2(y,z) := \exists x \varphi(x,y,z)$$

The required uniqueness follows from D being injective.

Exercise 2 Main ingredient is a formula that represents the function $(m,n) \mapsto m \bmod n$, like

$$\varphi(x,y,z) := z < y \wedge \exists u(m = u \cdot n + z).$$

Exercise 3 For instance for exponentiation, let $\varphi(x,y,z)$ express that

- $y = 0$ and $z = 1$, or
- $y > 0$ and there is a sequence $(a_i)_{0 \leqslant i \leqslant y}$ such that $a_0 = 1$ and $a_{i+1} = a_i \cdot x$ for $i < y$ and $z = a_y$.

The last condition is expressed with the help of the β-function, as in

$$\exists a \exists b \Big[\beta(a,b,0) = 1 \wedge \forall i\big(i < y \rightarrow \beta(a,b,i+1) = \beta(a,b,i) \cdot x\big) \wedge \beta(a,b,y) = z\Big],$$

where for instance $\beta(a,b,i+1) = \beta(a,b,i) \cdot x$ is shorthand for $\exists v \exists v'\big(\chi(a,b,i,v) \wedge \chi(a,b,i+1,v') \wedge v' = v \cdot x\big)$ for the formula χ representing the β-function.

Exercise 4 Let R_i be a register that is not used in the code of P. First, break any IF instruction that may point to itself into two IF instructions that may point to each other. Second, before every instruction of P, insert a line that pushes a symbol on R_i. Warning: make sure that relevant IF instructions would then point to the push instruction before the original instruction. Such a modified program either halts or has an aperiodic infinite run, since the number of symbols in register R_i keeps increasing.

Exercise 5 Note that in $\mathfrak{N}$ we may replace the ordering $<$ by its definition in terms of addition, according to $x < y \mapsto \exists z\big(\neg z = 0 \wedge z + x = y\big)$. (For $<$-atoms that involve complex terms one needs to make sure that the quantified variable z is distinct from all variables in those terms.) Let $\varphi \mapsto \varphi'$ be the syntactic translation that eliminates the use of $<$ in this manner.

Then $\varphi \in \mathrm{Th}(\mathfrak{N})$ iff $\varphi' \in \mathrm{Th}(\mathfrak{N})$. Putting $\mathrm{Th}(\mathfrak{N})' := \mathrm{Th}(\mathfrak{N}) \cap \mathrm{FO}_0(\{+,\cdot,0,1\})$, we see that $\mathrm{Th}(\mathfrak{N})'$ is undecidable.

To transfer this undecidability to $\mathrm{Th}(\mathfrak{Z})$ (even for $\mathfrak{Z} = (\mathbb{Z},+,\cdot,0,1)$ without the linear ordering!), let $\nu(x) = \exists z_1 \ldots \exists z_4 (x = \sum_{i=1}^{4} z_i^2)$. Then $\{m \in \mathbb{Z} \colon \mathfrak{Z} \models \nu[m]\} = \mathbb{N} \subseteq \mathbb{Z}$. For $\varphi \in \mathrm{FO}(\{+.\cdot,0,1\})$, we can now define φ^* inductively (as the so-called *relativisation* of φ to the subset defined by ν) such that for all $\bar{n}$ over $\mathbb{N}$, then:

$$(*) \qquad \mathfrak{N} \models \varphi[\bar{n}] \quad \text{iff} \quad \mathfrak{Z} \models \varphi^*[\bar{n}].$$

For atomic φ put $\varphi^* := \varphi$. $(*)$ holds, as $(\mathbb{N}, +^{\mathfrak{N}}, \cdot^{\mathfrak{N}}, 0^{\mathfrak{N}}, 1^{\mathfrak{N}}) \subseteq \mathfrak{Z}$ is a substructure.

Put $(\neg\varphi)^* := \neg\varphi^*$, $(\varphi_1 \circ \varphi_2)^* := \varphi_1^* \circ \varphi_2^*$, for $\circ = \wedge, \vee, \rightarrow, \leftrightarrow$. One checks that $(*)$ is preserved in these steps.

For the quantifier steps put $(\exists x \varphi)^* := \exists x \big(\nu(x) \wedge \varphi^*\big)$ and $(\forall x \varphi)^* := \forall x \big(\nu(x) \rightarrow \varphi^*\big)$. One checks that $(*)$ is again preserved.

For sentences in particular, $\mathfrak{N} \models \varphi$ iff $\mathfrak{Z} \models \varphi^*$.

It follows that $\mathrm{Th}(\mathfrak{Z})$ is undecidable, as $\mathrm{Th}(\mathfrak{N})$ is undecidable.

Equally, one can wrap the two translations into one, by substituting the definition of $<$ at the same time as relativising to $\mathbb{N}$ within $\mathbb{Z}$.