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Solution Hints for Exercises No.10X

Exercise 1 $\hat{\Phi}$ is recursive as the function $n \mapsto \psi_n$ enumerates it in lexicographic (in fact, length increasing) fashion (compare Exercise above).

$\Phi^{\vdash} = \hat{\Phi}^{\vdash}$ essentially follows from the finiteness property for $\vdash$:

If $\Phi \vdash \varphi$, the there is some n such that $\{\varphi_i : i \leqslant n\} \vdash \varphi$ and hence $\psi_n \vdash \varphi$ and also $\hat{\Phi} \vdash \varphi$; conversely, if $\hat{\Phi} \vdash \varphi$, then there is some n such that $\psi_n \vdash \varphi$ (since $\psi_n \vdash \psi_m$ for all $m \leqslant n$) and therefore $\{\varphi_i : i \leqslant n\} \vdash \varphi$, whence $\Phi \vdash \varphi$.

Exercise 2 Suppose first that φ^*_P is satisfiable, $\mathfrak{B} = (B, <^{\mathfrak{B}}, 0^{\mathfrak{B}}, W^{\mathfrak{B}}) \models \varphi^*_P$. Just as in the corresponding argument for Theorem 4.2.1 (Trakhtenbrot), we find that $\mathfrak{A}^n_P$ is isomorphic to the restriction of $\mathfrak{B}$ to its initial segment of length $n+1$. Therefore the part of the computation of P on $\square$ that is encoded in $\mathfrak{A}^n_P$ does not reach a halting configuration since $\mathfrak{B} \models \forall x \forall y \forall \bar{z}(Wxy\bar{z} \rightarrow \neg y = \underline{k})$. As this is true for every $n \in \mathbb{N}$, P does not terminate on $\square$.

If conversely P does not terminate on $\square$, then $\mathfrak{A}_P \models \varphi^*_P$.

Hence $\ulcorner P \urcorner \notin H$ iff $\varphi^*_P \in \mathrm{SAT}$.