Introduction to Mathematical Logic

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Solution Hints for Exercises No.1

Exercise 1 Substructures of

(1) $\mathfrak{A}_1 = (\mathbb{N}, \mathsf{S}, 0)$: only $\mathfrak{A}_1$ itself.

(2) $\mathfrak{A}_2 = (\mathbb{N}, \mathsf{S})$: all restrictions to subsets $\mathbb{N}_{\geqslant n} = \{m \in \mathbb{N} \colon m \geqslant n\}$ for $n \in \mathbb{N}$. However, any such is isomorphic to $\mathfrak{A}_2$ itself via the map $h \colon m \mapsto m - n$.

(3) $\mathfrak{A}_3 = (\mathbb{N}, \mathrm{graph}(\mathsf{S}))$: all restrictions to non-empty subsets of $\mathbb{N}$.

(4) $\mathfrak{A}_4 = \mathfrak{N} = (\mathbb{N}, +^{\mathfrak{N}}, \cdot^{\mathfrak{N}}, <^{\mathfrak{N}}, 0, 1)$: only $\mathfrak{A}_4$ itself.

(5) $\mathfrak{A}_5 = (\mathbb{N}, +^{\mathfrak{N}}, <^{\mathfrak{N}}, 0)$: all restrictions to subsets of $\mathbb{N}$ that contain 0 and that are closed under addition. Some of them are isomorphic to $\mathfrak{A}_5$, for instance $\mathfrak{A}_5 \restriction k\mathbb{N}$ for $k \neq 0$, and some are not, for instance with $\mathfrak{A}_5 \restriction (\{0\} \cup \mathbb{N}_{\geqslant 2})$ (check!).

Exercise 2 Composition of bijections from A to A is associative with neutral element id_A and with inverse maps for inverse elements. One checks that id_A is an automorphism of $\mathfrak{A}$; that the composition of any two automorphisms of $\mathfrak{A}$ is again an automorphism; and that the inverse of any automorphism is an automorphism. This shows that composition is an operation on $\mathrm{Aut}(\mathfrak{A})$, having a neutral element and inverses.

Exercise 3 (i) Closed under homomorphisms (check!), not closed under substructures, for instance $(\mathbb{N}, +, 0)$ is a substructure of $(\mathbb{Z}, +, 0)$ but not a group.

(ii) Closed under homomorphisms and closed under substructures.

(iii) Closed under homomorphisms (check by case distinction on whether $h(0) = h(1)$), not closed under substructures, for instance $\{\mathbb{N}, +, \cdot, 0, 1\}$ is a substructure of $\{\mathbb{R}, +, \cdot, 0, 1\}$ but not a field.

(iv) Closed under substructures, not closed under homomorphisms.
For instance $(\mathbb{N}, <^{\mathfrak{N}}) \xrightarrow{\text{hom}} (\mathbb{N}, U)$ where U is the universal binary relation on $\mathbb{N}$.

(v) Closed under substructures, not closed under homomorphisms.

Exercise 4 For countable signature σ:

(a) E.g., T_σ countable: ranking terms in T_σ according to (for instance) the number of applications of rule (T3) in their generation, show by induction on $n \in \mathbb{N}$ that the set of terms of rank n is countable. Then T_σ is countable as a countable union of countable sets.
Alternately, one can show that for a countable alphabet Σ, the language Σ^* of finite words over Σ is also countable. Both T_σ and $\mathrm{FO}(\sigma)$ are subsets of such a language.

(b) One may construct the closure of $A_0 \cup \{c^{\mathfrak{A}} \colon c \in \mathrm{const}(\sigma)\}$ under the $f^{\mathfrak{A}}$ for all $f \in \mathrm{fctn}(\sigma)$ algebraically (in countably many stages, each of which is countable).
Alternatively, look at the interpretation maps for terms, $\mathfrak{I}_\beta \colon T_\sigma \to A$ where $\beta \colon \mathrm{Var} \to A_0$ is a surjective assignment.
In this variant the image of the interpretation map for terms is the universe of a substructure $\mathfrak{B} \subseteq \mathfrak{A}$, contains A_0 and is countable.

Exercise 5 (a) Verify the closure conditions for constant and function symbols.

(b) For instance show by induction on $t \in T_\sigma(\emptyset)$ (or for $t \in T_\sigma$ with $\mathrm{var}(t) = \emptyset$) that for any homomorphism $h\colon \mathfrak{T}_\sigma(\emptyset) \xrightarrow{\text{hom}} \mathfrak{A}$ and any assignment $\beta\colon \mathrm{Var} \to A$, h must coincide with the interpretation function $\mathfrak{I}_\beta^{\mathfrak{A}}$ on $T_\sigma(\emptyset)$.

(c) Similar to part (b), but for any $\beta\colon \mathrm{Var} \to A$ that extends β_0.