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Winter 2012/13

Exercises No.8

Exercise 1 Discuss some of the following classes of structures and determine whether they are (Δ-)elementary or not:

(i) (dis-)connected graphs;
(ii) (non-)archimedean fields;
(iii) divisible groups;
(iv) (atomless) boolean algebras;
(v) (complete) real-closed ordered fields;
(vi) algebraically closed fields.

Exercise 2 Consider structures $\mathfrak{A} = (A, \mathsf{S}^{\mathfrak{A}}, 0^{\mathfrak{A}}, <^{\mathfrak{A}})$ of signature $\sigma = \{\mathsf{S}, 0, <\}$ with unary function S, constant 0 and binary relation $<$. Let φ be an FO(σ)-sentence saying that $<$ is a linear ordering (in the sense of $<$) with minimal element 0, that every element different from 0 is in the image of S, and that S maps every element to its direct successor w.r.t. $<$. Show that the following are equivalent for all $\mathfrak{A} \in \mathrm{Mod}(\varphi)$:

(i) all non-empty subsets $X \subseteq A$ have a $<^{\mathfrak{A}}$-minimal element.
(ii) for all subsets $X \subseteq A$: if $0^{\mathfrak{A}} \in X$ and X is closed under $\mathsf{S}^{\mathfrak{A}}$, then $X = A$.
(iii) for all $a \in A$, the $<^{\mathfrak{A}}$-interval $[0^{\mathfrak{A}}, a]$ is finite.
(iv) $\mathfrak{A} \simeq (\mathbb{N}, \mathsf{S}^{\mathfrak{N}}, 0^{\mathfrak{N}}, <^{\mathfrak{N}})$.

Exercise 3 Using the induction principle on ω show that the following are consequences of ZFC (even without (FOUND)):

(a) (i) $\forall x\big(x \in \omega \rightarrow$ "x transitive"$\big)$, which in particular implies transitivity of $\in$ over ω. (Recall that a set x is *transitive* if it satisfies $\forall y(y \in x \rightarrow y \subseteq x)$.)
(ii) $\forall x\big(x \in \omega \rightarrow \neg\, x \in x\big)$: irreflexivity of $\in$ over ω.
(iii) $\forall x \forall y\big((x \in \omega \wedge y \in \omega) \rightarrow (x \in y \vee x = y \vee y \in x)\big)$ (trichotomy for $\in$ over ω). Hint: show that the set $\{x \in \omega : \forall y(y \in \omega \rightarrow (x \in y \vee x = y \vee y \in x))\}$ is inductive; it may be useful to establish first that, for all $z, y \in \omega$: $z \in y \Rightarrow (\mathsf{S}z \in y \vee \mathsf{S}z = y)$.

(b) Prove from the characterisation of ω as the minimal inductive set: $\in$ well-orders ω.

Exercise 4

(a) Show that (FOUND) rules out the existence of $\in$-cycles $x_0 \in x_1 \in \cdots \in x_{n-1} \in x_0$, for any $0 < n \in \mathbb{N}$. Use this to show that ZFC $\models (x \cup \{x\} = y \cup \{y\}) \rightarrow x = y$.
(b) Show that $(x \cup \{x\} = y \cup \{y\}) \rightarrow x = y$ even without (FOUND).
(c) Justify the following principle of $\in$-induction on the basis of (FOUND). Suppose that for $\varphi(x) \in \mathrm{FO}(\{\in\})$,

$$\mathrm{ZFC} \models \forall x\Big(\forall z(z \in x \rightarrow \varphi(z)) \rightarrow \varphi(x)\Big),$$

where we write $\varphi(z)$ for $\varphi\frac{z}{x}$. Show that then ZFC $\models \forall x($"x transitive" $\rightarrow \varphi(x))$.

Hint: assume to the contrary that $\neg\varphi(x)$ for some transitive x and look at an $\in$-minimal element of $\big\{z \in x \colon \neg\varphi(z)\big\}$ to derive a contradiction.

Exercise 5 [Extra] Sketch formulations of the following mathematical principles in the framework of ZFC (and discuss in outline how they can be proved in ZFC):

(a) For any non-empty set (family) x of sets, there exists a set representing the product over the sets $(y)_{y \in x}$; the product of a non-empty family of non-empty sets is non-empty.

(b) The power set of any set is of strictly larger cardinality than the set itself.

(c) The set of all sequences of natural numbers exists, and is not countable.