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Exercises No.7

Exercise 1 [well-orderings]
Consider the signature $\sigma = \{<\}$ with a binary relation symbol $<$. A σ-structure $\mathfrak{A} = (A, <^{\mathfrak{A}})$ is called a *well-ordering* (or $<^{\mathfrak{A}}$ is said to well-order A) if $<^{\mathfrak{A}}$ is a strict linear ordering of the universe A such that every non-empty subset $A' \subseteq A$ has a $<^{\mathfrak{A}}$-minimal element.

(a) Show that a linear ordering $\mathfrak{A} = (A, <^{\mathfrak{A}})$ is a well-ordering if, and only if, there is no infinite descending sequence w.r.t. $<^{\mathfrak{A}}$.

(b) Show that the class of all well-orderings is not Δ-elementary.

Exercise 2 we look at two consequences of the compactness theorem for FO. The second holds the key also to the link with the topological meaning: FO compactness *is* a finite-sub-cover property for a suitable topological space of FO-theories.

(a) Let $\mathcal{C}$ be a class of σ-structures, $\overline{\mathcal{C}}$ its complement, i.e. the class of all σ-structures not in $\mathcal{C}$. Show that if both $\mathcal{C}$ and $\overline{\mathcal{C}}$ are Δ-elementary, then they must both be elementary. [Hint: look at the union of the two sets of sentences that axiomatise $\mathcal{C}$ and $\overline{\mathcal{C}}$, respectively.]

(b) Suppose $\Psi \subseteq \mathrm{FO}_0(\sigma)$ is a set of sentences such that every σ-structure satisfies at least one sentence $\psi \in \Psi$. Then there is a finite $\Psi_0 \subseteq \Psi$ with the same property, i.e. such that $\models \bigvee_{\psi \in \Psi_0} \psi$. [Hint: consider the set $\{\neg\psi : \psi \in \Psi\}$.]
Extra: interpret this as a topological notion of compactness; abstract a suitable topological space and discuss some of its further topological properties.

Exercise 3 Let $\mathfrak{A}$ be a σ-structure, $\Phi = \{\varphi_i(x) : i \in \mathbb{N}\} \subseteq \mathrm{FO}(\sigma)$ a set of σ-formulae in a single free variable x. Suppose that for all $n \in \mathbb{N}$, $\mathfrak{A} \models \exists x \bigwedge_{i \leqslant n} \varphi_i$.

(a) Give an example that there need not be any element $a \in A$ s.t. for all i, $\mathfrak{A} \models \varphi_i[a]$.

(b) Show that there is $\mathfrak{A}' \equiv \mathfrak{A}$ with $a \in A'$ s.t. for all i, $\mathfrak{A}' \models \varphi_i[a]$.

Exercise 4 Consider the expansion of the standard structure $\mathfrak{R} = (\mathbb{R}, +^{\mathfrak{R}}, \cdot^{\mathfrak{R}}, 0^{\mathfrak{R}}, 1^{\mathfrak{R}}, <^{\mathfrak{R}})$ by a unary function $F : \mathbb{R} \to \mathbb{R}$ with $F(0) = 0$ as a $(\sigma_{\mathrm{ar}} \cup \{f\})$-structure. Let a non-standard version $(\mathfrak{R}^*, F^*)$ be obtained, with infinitesimal elements, exactly as for $\mathfrak{R}^*$. Let $N(0)$ be the infinitesimal neighbourhood of 0 in $(\mathfrak{R}^*, F^*)$, which consists of all those elements $u \in \mathbb{R}^*$ for which $-1 < \underline{n}u < 1$ for all $n \in \mathbb{N}$ (we write just $<$ for $<^{\mathfrak{R}^*}$, $\underline{n}$ for $\underline{n}^{\mathfrak{R}^*}$ etc).
Show that F is continuous at $x = 0$ iff $F^*[N(0)] \subseteq N(0)$.

Hint: use a characterisation of continuity in the standard model that works with ε, δ of the form $1/n$ for $1 \leqslant n \in \mathbb{N}$; then, for instance, $F[(-1/n, 1/n)] \subseteq (-1/m, 1/m)$ for fixed $n, m \in \mathbb{N}$ can be expressed in $\mathrm{FO}(\sigma_{\mathrm{ar}} \cup \{f\})$; so, for *fixed* $n, m \in \mathbb{N}$, $(\mathfrak{R}^*, F^*) \equiv (\mathfrak{R}, F)$ can be used to transfer such assertions between $(\mathfrak{R}, F)$ to $(\mathfrak{R}^*, F^*)$.

Extra: can you think of other analytic limit phenomena that may have simpler limit-free descriptions in the non-standard framework?