Introduction to Mathematical Logic

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Exercises No.6

Exercise 1 A formula is *equality-free* if it contains no term equations. A formula is *universal* if it is in negation normal form without existential quantifiers, i.e., built from atomic and negated atomic formulae by means of $\vee$, $\wedge$ (where $\varphi_1 \wedge \varphi_2$ is shorthand for $\neg(\neg\varphi_1 \vee \neg\varphi_2)$) and $\forall$ (where $\forall x\varphi$ is shorthand for $\neg\exists x\neg\varphi$).

(a) Show that any satisfiable set $\Phi \subseteq \mathrm{FO}(\sigma)$ of equality-free universal formulae has a model obtained from an expansion of $\mathfrak{T}_\sigma$ by suitable interpretations for the relation symbols in σ.
Hint: think of extracting some interpretation of $R \in \sigma$ over T_σ from any given $\mathfrak{A}, \beta \models \Phi$.

(b) Show that for any set $\Phi \subseteq \mathrm{FO}(\sigma)$ of equality-free universal formulae, which is maximally consistent w.r.t. equality-free universal formulae, the analogue of the Henkin model for Φ based on T_σ (without passage to a quotient) satisfies $\mathfrak{H}, \beta^{\mathfrak{H}} \models \Phi$.
Hint: show by syntactic induction that for all equality-free universal formulae φ,

$$\Phi \vdash \varphi \;\Rightarrow\; \mathfrak{H}, \beta^{\mathfrak{H}} \models \varphi.$$

Exercise 2

(a) Let $\sigma' = \sigma \cup \{c\}$ where the constant symbol c is not in σ, $\Phi \subseteq \mathrm{FO}(\sigma) \subseteq \mathrm{FO}(\sigma')$, $\varphi \in \mathrm{FO}(\sigma)$.

(i) Show that $\mathrm{cons}_\sigma(\Phi) \Rightarrow \mathrm{cons}_{\sigma'}(\Phi)$. [Hint: assuming that Φ were inconsistent w.r.t. σ', look at a derivation of a contradiction; using the finiteness of this derivation, find a way to transform it into a derivation of a contradiction over σ.]

(ii) Show that $\mathrm{cons}_\sigma(\Phi) \Rightarrow \mathrm{cons}_{\sigma'}(\Phi \cup \{\exists x\varphi \rightarrow \varphi\frac{c}{x}\})$.

(iii) For countable σ, use (a) and (b) to find an extension of Φ that has witnesses and that only uses new constant symbols as witnesses. [NB: with new constants we also produce new formulae, hence require new witnesses; this suggests to use a chain of extensions.]

(b) Assuming completeness, give a semantic argument that for $\sigma \subseteq \sigma'$ and $\Phi \subseteq \mathrm{FO}(\sigma)$, $\mathrm{cons}_\sigma(\Phi) \;\Rightarrow\; \mathrm{cons}_{\sigma'}(\Phi)$.

Exercise 3 [witnesses and Skolemisation]
Consider the following *Skolem theory* $\mathrm{Sk}_0(\sigma)$ for a signature σ:

$$\mathrm{Sk}_0(\sigma) := \big\{\forall \mathbf{x}(\exists y\varphi(\mathbf{x}, y) \rightarrow \varphi\tfrac{f_{\varphi\mathbf{x}y}\mathbf{x}}{y})\colon \varphi \in \mathrm{FO}(\sigma)\big\},$$

for new function symbols $f_{\varphi\mathbf{x}y} \notin \sigma$, where the arity of $f_{\varphi\mathbf{x}y}$ matches the arity of the tuple $\mathbf{x}$.
[Idea: $f_{\varphi\mathbf{x}y}$ serves as a *Skolem function* that provides witnesses for $\exists y\varphi$ if there are any such.]
Show that

(a) any σ-structure can be expanded to a model of $\mathrm{Sk}_0(\sigma)$.

(b) every $\varphi \in \mathrm{FO}(\sigma)$ is equivalent under $\mathrm{Sk}_0(\sigma)$ to a formula (in the extended signature) that is purely universal (generated from atomic/negated atomic formulae by $\vee$, $\wedge$, $\forall$).

Exercise 4 [extra: excursion on uses of Zorn's lemma]
Apply Zorn's lemma to suitable partial orderings in order to show that it implies the following (in fact it is equivalent to each of these, relative to the remaining axioms of standard Zermelo–Fraenkel set theory, ZF):

(a) The *axiom of choice*: for every family $(A_i)_{i \in I}$ of non-empty sets A_i (indexed by any set I), there is a choice function, i.e., a function $f \colon I \to \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$.

(b) The Cartesian product of any family of non-empty sets is non-empty.

(c) The *well-ordering principle*: every set A can be well-ordered, i.e., there exists a binary relation $<^{\mathfrak{A}} \subseteq A \times A$ such that $(A, <^{\mathfrak{A}})$ is a well-ordering.

Hint: think of partially ordered sets of suitable 'approximations' to the desired object, such that maximality of an approximation means that it is as desired.