Introduction to Mathematical Logic

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Exercises No.4

Exercise 1
Show that the following propositional rules are correct and in fact derivable rules of $\mathcal{S}$.

(a)
$$\frac{\begin{array}{ll}\Gamma & (\varphi \to \psi)\\ \Gamma & \varphi\end{array}}{\Gamma\ \psi}\ \text{(modus ponens)} \qquad \frac{\Gamma\ \varphi\ \neg\psi}{\Gamma\ \psi\ \neg\varphi}\ \text{(contr 2)}$$

For the first one note that $(\varphi \to \psi)$ is treated as an abbreviation for $(\neg\varphi \vee \psi)$.

(b)
$$\frac{\Gamma\ \neg\neg\varphi}{\Gamma\ \varphi}\ \text{(NN 1)} \qquad \frac{\Gamma\ \varphi}{\Gamma\ \neg\neg\varphi}\ \text{(NN 2)}$$

Note that the variants of these double negation rules in the form of sequents $\Gamma\ \neg\neg\varphi\ \varphi$ and $\Gamma\ \varphi\ \neg\neg\varphi$ are easily derivable from the corresponding contrapositive rules.

Exercise 2 Discuss rules for the introduction of propositional connectives $\wedge$ or $\to$ in the antecedent/succedent. What are appropriate criteria?

Exercise 3 Show that the following rule is correct and that it is derivable in $\mathcal{S}$:

$$\frac{\Gamma\ \neg\exists x\neg\varphi}{\Gamma\ \varphi\frac{t}{x}}\ (\forall\text{-instantiation}).$$

Exercise 4 Give a formal proof, through an $\mathcal{S}$-derivation of the following (obviously valid) 'theorem', in a signature σ with a constant symbol c and unary function symbol f:

$$\neg\exists x\neg\exists y\ x = fy \quad \vdash \quad \exists x\ c = ffx.$$

Note that the left hand sentence says that f is surjective. Hint: one may try to use a sequent of the form $c = fy\ y = fz\ c = ffz$ as an intermediate step, and successively quantify variables in the right order with appropriate $\exists$ introduction rules. Note also that an application of the derived rule from exercise 3 above allows one to make use of instantiations of the (universal) sentence from the left hand side of the desired derivation, once one has a copy of it on the right.

examples of derived rules

(Ctr')
$$\begin{array}{c} \Gamma \ \varphi \\ \Gamma \neg\varphi \\ \hline \Gamma \ \psi \end{array}$$

(chain)
$$\begin{array}{c} \Gamma \quad \varphi \\ \Gamma \quad \varphi \quad \psi \\ \hline \Gamma \quad \psi \end{array}$$

(contr 1)
$$\begin{array}{c} \Gamma \ \neg\varphi \ \psi \\ \hline \Gamma \ \neg\psi \ \varphi \end{array}$$

(contr 2)
$$\begin{array}{c} \Gamma \ \varphi \ \neg\psi \\ \hline \Gamma \ \psi \ \neg\varphi \end{array}$$

[plus two more]

(NN 1)
$$\begin{array}{c} \Gamma \ \neg\neg\varphi \\ \hline \Gamma \ \varphi \end{array}$$

(NN 2)
$$\begin{array}{c} \Gamma \ \varphi \\ \hline \Gamma \ \neg\neg\varphi \end{array}$$

(impl 1)
$$\begin{array}{c} \Gamma \quad \neg\varphi \\ \Gamma \quad (\varphi \vee \psi) \\ \hline \Gamma \ \psi \end{array}$$

(impl 2)
$$\begin{array}{c} \Gamma \quad \varphi \\ \Gamma \quad (\varphi \to \psi) \\ \hline \Gamma \ \psi \end{array}$$

"modus ponens"

(∀S)
$$\begin{array}{c} \Gamma \quad \varphi\frac{y}{x} \\ \hline \Gamma \ \ \forall x\varphi \end{array}$$

y not in free$(\Gamma, \forall x\varphi)$

(∀ inst)
$$\begin{array}{c} \Gamma \quad \forall x\varphi \\ \hline \Gamma \quad \varphi\frac{t}{x} \end{array}$$

(symm =)
$$\begin{array}{c} \Gamma \ t = t' \\ \hline \Gamma \ t' = t \end{array}$$

(trans =)
$$\begin{array}{c} \Gamma \quad t = t' \\ \Gamma \quad t' = t'' \\ \hline \Gamma \quad t = t'' \end{array}$$

(congr =, $\overset{(n)}{f}$)
$$\begin{array}{c} \Gamma \quad t_1 = t'_1 \\ \vdots \\ \Gamma \quad t_n = t'_n \\ \hline \Gamma \ f t_1 \dots t_n = f t'_1 \dots t'_n \end{array}$$

(congr =, $\overset{(n)}{R}$)
$$\begin{array}{c} \Gamma \quad t_1 = t'_1 \\ \vdots \\ \Gamma \quad t_n = t'_n \\ \Gamma \quad R t_1 \dots t_n \\ \hline \Gamma \quad R t'_1 \dots t'_n \end{array}$$