Introduction to Mathematical Logic

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Exercises No.3

Exercise 1 (a) How would you model vector spaces as σ-structures (with certain constraints) for suitable choices of a signature σ?

(i) Vector spaces V over a field $\mathbb{F}$ (where both $\mathbb{F}$ and V vary).

(ii) Vector spaces V over a fixed field $\mathbb{F}$ (in particular, a fixed finite field like $\mathbb{F}_p$).

(b) Over the class of expansions of the standard structure $\mathfrak{R} = (\mathbb{R}, +, \cdot, 0, 1, <)$ to σ-structures, for $\sigma := \sigma_{\text{ar}} \cup \{f\}$ with a unary function symbol f, express the following properties in $\text{FO}(\sigma)$:

(i) f is continuous at 0.

(ii) f is uniformly continuous.

(iii) f is differentiable at 0.

Remark: Try to modularise your attempts by first formalising useful subformulae that can be re-used.

Exercise 2 A formula $\varphi \in \text{FO}(\sigma)$ is said to be in *negation normal form* (nnf) if it is built with conjunction, disjunction and existential and universal quantification from atoms and negated atoms. Define, inductively on $\text{FO}(\sigma)$, a map from $\text{FO}(\sigma)$ to the subset of nnf-formulae in $\text{FO}(\sigma)$, $\varphi \mapsto \text{nnf}(\varphi)$ such that $\varphi \equiv \text{nnf}(\varphi)$. Prove by induction that your map is as required.

Exercise 3 Consider formulae $\varphi \in \text{FO}(\sigma)$ in negation normal form (see above). With a fixed σ-structure $\mathfrak{A}$ associate the following two-person game between a *Refuter* R and a *Verifier* V (also in the literature: Adam and Eve, Abelard and Eloise, $\forall$ and $\exists$). Positions of the game are pairs (φ, β) for nnf formulae φ and assignments β in $\mathfrak{A}$ (partial assignments to free(φ) would also suffice). Who is to move in position (φ, β), and the possible such moves, depends on φ:

- if $\varphi = (\varphi_1 \vee \varphi_2)$: V to move to one of (φ_1, β) or (φ_2, β) (her choice).
- if $\varphi = (\varphi_1 \wedge \varphi_2)$: R to move to one of (φ_1, β) or (φ_2, β) (his choice).
- if $\varphi = \exists x \varphi'$, V is to move to a position $(\varphi', \beta\frac{a}{x})$ for an element $a \in A$ of her choice.
- if $\varphi = \forall x \varphi'$, R is to move to a position $(\varphi', \beta\frac{a}{x})$ for an element $a \in A$ of his choice.

In any other position (φ, β), φ is either atomic or negated atomic. These are the final positions in the game, in which V wins if $(\mathfrak{A}, \beta) \models \varphi$ while R wins if $(\mathfrak{A}, \beta) \not\models \varphi$ (determined in terms of term equalties and membership in relations). Show by induction on nnf formulae φ that

$$\text{V has a winning strategy in position } (\varphi, \beta) \quad \text{iff} \quad (\mathfrak{A}, \beta) \models \varphi,$$

$$\text{R has a winning strategy in position } (\varphi, \beta) \quad \text{iff} \quad (\mathfrak{A}, \beta) \not\models \varphi.$$

[A winning strategy requires a choice of own moves in response to any choice of moves from the opponent such that a win in the game is guaranteed; as formulae get shorter in each move, the game terminates in a finite number of rounds that can be bounded in terms of the current formula.]

Extra: What is the right version of the game if negation is freely allowed?