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Exercises No.2

Exercise 1 (a) Prove the following assertion for all $t \in T_\sigma$ by (syntactic) induction based on the calculus that generates T_σ: no proper prefix of t is an element of T_σ, and t is not a proper prefix of any $t' \in T_\sigma$.

(b) For $V \subseteq \mathrm{Var}$ we defined $T_\sigma(V) := \{t \in T_\sigma : \mathrm{var}(t) \subseteq V\}$. Alternatively, one could use a modified calculus to generate just the terms with variable symbols from V. Do so, and show that the results agree.

Exercise 2 (Ex. 1.5 again, now that we have the prerequisites.)

Let σ be a signature consisting of only function and constant symbols, and with $\mathrm{const}(\sigma) \neq \emptyset$. Let $\mathfrak{T}_\sigma$ be the *free or term σ-structure* with universe T_σ. With a subset $V_0 \subseteq \mathrm{Var}$ associate the set of terms $T_\sigma(V_0) := \{t \in T_\sigma : \mathrm{var}(t) \subseteq V_0\}$.

(a) Show that $T_\sigma(V_0)$ is the universe of a substructure $\mathfrak{T}_\sigma(V_0) := \mathfrak{T}_\sigma \restriction T_\sigma(V_0) \subseteq \mathfrak{T}_\sigma$.

(b) Show that for any σ-structure $\mathfrak{A}$ there is a unique homomorphism $h \colon \mathfrak{T}_\sigma(\emptyset) \xrightarrow{\text{hom}} \mathfrak{A}$.

(c) Let $\beta_0 \colon V_0 \to A$ be a *partial assignment* in the σ-structure $\mathfrak{A}$. Show that there is a unique homomorphism $h \colon \mathfrak{T}_\sigma(V_0) \xrightarrow{\text{hom}} \mathfrak{A}$ that extends β_0 [in fact: the restriction of the interpretation function of terms for any assignment that extends β_0].

Exercise 3 Consider $\sigma = \{R\}$ with a binary relation symbol R. Show that the following classes of σ-structures are elementary, by providing appropriate $\mathrm{FO}(\sigma)$ sentences.

(i) $\mathcal{C}_1$ the class of all $\mathfrak{A} = (A, R^{\mathfrak{A}})$ s.t. $R^{\mathfrak{A}}$ linearly orders A without maximal element.

(ii) $\mathcal{C}_2$ the class of $\mathfrak{A} = (A, R^{\mathfrak{A}})$ in which $R^{\mathfrak{A}}$ is the graph of a unary function $f \colon A \to A$ that is injective but not surjective.

(iii) $\mathcal{C}_3$ the class of $\mathfrak{A} = (A, R^{\mathfrak{A}})$ s.t. $R^{\mathfrak{A}}$ is an equivalence relation on A with at least two equivalence classes.

(iv) $\mathcal{C}_4$ the class of all $\mathfrak{A} = (A, R^{\mathfrak{A}}) \simeq (\{1,2,3\}, \{(1,2),(2,3),(3,1)\})$.

In at least one case, check the adequacy of your formalisation by exhibiting the semantics according to the inductive definition of the satisfaction relation.

Exercise 4 Call a formula $\varphi \in \mathrm{FO}(\sigma)$ *positive* if it is generated from atomic formulae by conjunctions, disjunctions and quantifications alone (i.e., no use of (F3), or of $\to$ or $\leftrightarrow$ in (F4)).

Let $h \colon \mathfrak{A} \xrightarrow{\text{hom}} \mathfrak{B}$ be a surjective homomorphism. For an assignment β in $\mathfrak{A}$, $h \circ \beta$ is an assignment in $\mathfrak{B}$ and every assignment in $\mathfrak{B}$ may be thus obtained. Show the following by syntactic induction for all positive formulae $\varphi \in \mathrm{FO}(\sigma)$:

$$(\mathfrak{A}, \beta) \models \varphi \quad \Rightarrow \quad (\mathfrak{B}, h \circ \beta) \models \varphi.$$

Give examples to show that surjectivity of h and positivity of φ are both necessary.

Exercise 5 Prove the following, for arbitrary $\varphi, \psi \in \mathrm{FO}(\sigma)$ and $x, y \in \mathrm{Var}$:

(i) $\forall x \forall y \varphi \equiv \forall y \forall x \varphi$.

(ii) $\exists x \forall y \varphi \models \forall y \exists x \varphi$.

(iii) if $x \notin \mathrm{free}(\varphi)$: $(\varphi \land \forall x \psi) \equiv \forall x(\varphi \land \psi)$.