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Exercises No.13

Exercise 1

(a) Give an example of structures $\mathfrak{A}$ and $\mathfrak{B}$ in an infinite relational vocabulary for which $\mathfrak{A} \equiv \mathfrak{B}$ but not $\mathfrak{A} \simeq_{\text{fin}} \mathfrak{B}$.

(b) For structures in any relational vocabulary, argue that $\simeq_{\text{fin}}$ coincides with $\simeq$ if at least one of the structures involved is finite.

(c) Give examples that $\simeq_{\text{fin}}$ and $\simeq_{\text{part}}$ are distinct.

Exercise 2 Consider two discrete linear orderings $\mathfrak{A} = (A, <^{\mathfrak{A}})$ and $\mathfrak{B} = (B, <^{\mathfrak{B}})$ with first and last elements. Here *discrete* means that each element apart form the last (first) has an immediate successor (predecessor). For two elements $x \leqslant y$ in either ordering and $\ell \in \mathbb{N}$ consider the following truncated distance:

$$d^{\ell}(x,y) := \begin{cases} |\{z \colon x \leqslant z < y\}| & \text{if } |\{z \colon x \leqslant z < y\}| < 2^{\ell} \\ \infty & \text{else} \end{cases}$$

Let $a_{\text{min}}/b_{\text{min}}$ the first elements, $a_{\text{max}}/b_{\text{max}}$ the last elements of $\mathfrak{A}/\mathfrak{B}$, respectively. For $\ell \in \mathbb{N}$ let $\dot{I}_{\ell} \subseteq \text{Part}(\mathfrak{A}, \mathfrak{B})$ be the set of all $p \colon \bar{a} \mapsto \bar{b}$ for any pair $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$ of strictly increasing n-tuples, $n \geqslant 2$, of the form

$$\begin{array}{l} a_{\text{min}} = a_1 <^{\mathfrak{A}} \cdots <^{\mathfrak{A}} a_n = a_{\text{max}} \\ b_{\text{min}} = b_1 <^{\mathfrak{B}} \cdots <^{\mathfrak{B}} b_n = b_{\text{max}} \end{array} \qquad \text{such that } d^{\ell}(a_i, a_{i+1}) = d^{\ell}(b_i, b_{i+1}) \text{ for } 1 \leqslant i < n.$$

Show that $(\dot{I}_{\ell})_{\ell \in \mathbb{N}}$ is a back-and-forth system.

Use this insight to show that

(a) the class of finite linear orderings of even length is not FO-definable within the class of all finite linear orderings.

(b) the ordering of $\mathbb{N}$ is elementarily equivalent to that of the naturals with a copy of the integers appended on the right.

Exercise 3 Show that the FO($<$)-sentence asserting that $<$ is a dense linear ordering without end points has, up to isomorphism, just one countable model (viz., the ordering of the rational numbers).

Exercise 4 Discuss the changes required in the Ehrenfeucht–Fraïssé Theorem for finite signatures that may have constants and function symbols.

Hint: constants pose no real problem at all (why?); for functions, one may either replace quantifier rank by a more fine-grained rank that also takes into account the complexity of terms; or one may use a normal form for the use of terms that forces the quantifier rank to increase accordingly with the nesting of terms.

Exercise 5 [(extra, for next week)]
Let FO_∞ be the infinitary variant of first-order logic which allows conjunctions and disjunctions over arbitrary (in particular also infinite) sets of formulae in finitely many free variables. The semantics of the new formulae is the obvious one, with, e.g.,

$$\mathfrak{A}, \mathbf{a} \models \bigwedge_{i \in I} \varphi_i(\mathbf{x}) \quad \text{if} \quad \mathfrak{A}, \mathbf{a} \models \varphi_i(\mathbf{x}) \text{ for all } i \in I.$$

The quantifier rank of formulae in FO_∞ is naturally defined as an ordinal-valued function, with, e.g., the new clause $\mathrm{qr}(\bigwedge_{i \in I} \varphi_i) := \sup\{\mathrm{qr}(\varphi_i) \colon i \in I\}$. Show the following for structures $\mathfrak{A}, \mathfrak{B}$ in a relational signature.

Karp's Theorem:
$\mathfrak{A}, \mathbf{a} \simeq_{\mathrm{part}} \mathfrak{B}, \mathbf{b}$ if, and only if, $\mathfrak{A}, \mathbf{a}$ and $\mathfrak{B}, \mathbf{b}$ are indistinguishable in FO_∞.