Introduction to Mathematical Logic

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Exercises No.12$\frac{1}{2}$ (extra)

These extras concern abstract reasoning on the basis of Löb's 'axiomatic' characterisation of the natural internal notion of 'provability' (e.g., in PA). One of these exercises – Löb's Theorem, Exercise 3 below – was already included as Exercise 4 in the regular sheet no. 12.

Exercise 1 [warm-up]
Show that, in the presence of (L1), the axiom (L2) on 'internal modus ponens' also implies that 'provability distributes over implication' in the sense that

$$(\ddagger) \qquad \Phi \vdash \alpha \rightarrow \beta \quad \Rightarrow \quad \Phi \vdash \mathrm{prov}_\Phi(\ulcorner \underline{\alpha} \urcorner) \rightarrow \mathrm{prov}_\Phi(\ulcorner \underline{\beta} \urcorner).$$

Remark: we used ($\ddagger$) in the proof of the second incompleteness thm on the basis of (L1)–(L3).

Exercise 2 [Kreisel]
Show that under (L1)–(L3) any fixpoint sentence φ with $\Phi \vdash \varphi \leftrightarrow \neg\mathrm{prov}_\Phi(\ulcorner \underline{\varphi} \urcorner)$ satisfies

$$\Phi \vdash \varphi \leftrightarrow \mathrm{cons}_\Phi,$$

where $\mathrm{cons}_\Phi = \neg\mathrm{prov}_\Phi(\ulcorner \underline{\bot} \urcorner)$ for any $\bot$ such that $\vdash \neg\bot$. It follows that any two fixpoint sentences for $\psi(x) := \neg\mathrm{prov}_\Phi(x)$ are provably equivalent under Φ, as are the consistency statements based on any choice of contradiction.

Exercise 3 (Löb's Theorem)
Use (L1), (L2), (L3) and the existence of a fixpoint formula φ for $\psi(x) := \mathrm{prov}_\Phi(x) \rightarrow \eta$ to show that

$$\Phi \vdash \mathrm{prov}_\Phi(\ulcorner \underline{\eta} \urcorner) \rightarrow \eta \quad \Rightarrow \quad \Phi \vdash \eta.$$

NB: this implies that any fixpoint sentence for $\psi(x) := \mathrm{prov}_\Phi(x)$ *is* provable from Φ.

Hint: from $\Phi \vdash \varphi \rightarrow \big(\mathrm{prov}_\Phi(\ulcorner \underline{\varphi} \urcorner) \rightarrow \eta\big)$ obtain, by applications of (L1) and (L2), that

$$\Phi \vdash \mathrm{prov}_\Phi(\ulcorner \underline{\varphi} \urcorner) \rightarrow \big(\mathrm{prov}_\Phi(\ulcorner \underline{\mathrm{prov}_\Phi(\ulcorner \underline{\varphi} \urcorner)} \urcorner) \rightarrow \mathrm{prov}_\Phi(\ulcorner \underline{\eta} \urcorner)\big).$$

Using the assumption on η and (L1)–(L3), one obtains that $\Phi \vdash \mathrm{prov}_\Phi(\ulcorner \underline{\varphi} \urcorner) \rightarrow \eta$, that $\Phi \vdash \varphi$, $\Phi \vdash \mathrm{prov}_\Phi(\ulcorner \underline{\varphi} \urcorner)$, and finally $\Phi \vdash \eta$.

Exercise 4 [easy corollary/bonus material]
Use Löb's Theorem for $\eta := \bot$ to derive the second incompleteness theorem.