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Exercises No.12

The first three exercises refer to the weak first-order arithmetical theory Q axiomatised by

$$\begin{array}{ll} (\mathrm{Q1}) & \forall x \neg\, 0 = x+1 \\ (\mathrm{Q2}) & \forall x \forall y (x+1 = y+1 \rightarrow x = y) \\ (\mathrm{Q3}) & \forall x (\neg\, x = 0 \rightarrow \exists y\, x = y+1) \\ (\mathrm{Q4}) & \forall x (x+0 = x) \\ (\mathrm{Q5}) & \forall x \forall y (x + (y+1) = (x+y)+1) \\ (\mathrm{Q6}) & \forall x (x \cdot 0 = 0) \\ (\mathrm{Q7}) & \forall x \forall y (x \cdot (y+1) = (x \cdot y) + x) \end{array}$$

Exercise 1 Let $\underline{0} := 0$ and $\underline{n+1} := \underline{n} + 1$ be the usual representation.

(a) Show (by induction on n) that for all $n < m$ in $\mathbb{N}$: $Q \vdash \neg\, \underline{n} = \underline{m}$.

(b) Provide a representation of the addition function, by showing that the obvious definition using the given function $+$ works as a representation.

(c) Show that the composition $h = f \circ g$ of two functions $f, g \colon \mathbb{N} \rightarrow \mathbb{N}$ that are represented by φ_f and φ_g, respectively, is again representable.

Exercise 2 Q has no axioms concerning a linear ordering, but $\varphi_{\leqslant}(x, y) := \exists z (z + x = y)$ defines the usual reflexive linear ordering $\leqslant^{\mathfrak{N}}$ in the standard model $\mathfrak{N}$.

(a) Show (by induction on n) that $Q \vdash \forall x (\varphi_{\leqslant}(x, \underline{n}) \rightarrow (x = \underline{0} \vee x = \underline{1} \vee \ldots \vee x = \underline{n}))$.

(b) Show that $\varphi_{\leqslant}$ represents the relation $\leqslant^{\mathfrak{N}}$.

(c) Show that, if $\mathfrak{A} = (A, +^{\mathfrak{A}}, \cdot^{\mathfrak{A}}, 0^{\mathfrak{A}}, 1^{\mathfrak{A}}) \models Q$, the set $B := \{\underline{n}^{\mathfrak{A}} \colon n \in \mathbb{N}\}$ is the universe of an induced substructure $\mathfrak{B} := \mathfrak{A}{\restriction} B$ that is isomorphic to (the $\{+, \cdot, 0, 1\}$-reduct of) $\mathfrak{N}$, and ordered by $\varphi_{\leqslant}$ like $\mathfrak{N}$.

(d) Does Q imply that $\varphi_{\leqslant}$ defines a linear ordering of the universe? Discuss the potential differences between the relation defined by $\varphi_{\leqslant}$ and by $\varphi'_{\leqslant}(x, y) := \exists z (x + z = y)$.
Hint: try to find a non-standard interpretation of $+$ over some universe that is not fully covered by the terms $\underline{n}$. E.g., look at an extension of the standard model $\mathfrak{N}$ by an "infinite" part structured like $\mathbb{Z}$ w.r.t. the successor $z \mapsto z+1$.
Does Q force $+$ to be commutative?

Exercise 3 [extra] Provide a proof sketch that Q admits representations for every *total* recursive function, i.e., for every total function of the form $f \colon \mathbb{N}^n \rightarrow \mathbb{N}$ computed by some R-program P. For this, consider carefully how one needs to adapt and modify the formula φ_P we used in our proof of Tarski's theorem for the reduction of the question whether $P \in \mathrm{H}$ to whether $\varphi_P \in \mathrm{Th}(\mathfrak{N})$, to obtain a formula $\varphi(\mathbf{x}, z)$ that represents f not only w.r.t. to $\mathrm{Th}(\mathfrak{N})$ but even w.r.t. the much weaker Q.

NB: a straightforward modification that takes care of input/output will be good enough to achieve representation in $\mathrm{Th}(\mathfrak{N})$, and that $Q \vdash \varphi(\underline{\mathbf{m}}, \underline{f(\mathbf{m})})$, and even $Q \vdash \varphi(\underline{\mathbf{m}}, \underline{n})$ precisely for $n = f(\mathbf{m})$ (why?); but it will in general fail to satisfy the stronger functionality requirement $Q \vdash \forall z \big(\varphi(\underline{\mathbf{m}}, z) \rightarrow z = \underline{f(\mathbf{m})}\big)$ (why?; what can be done about it?; and how does totality of f matter?). Think of what can go wrong in non-standard models of Q.

Exercise 4 (Löb's Theorem) Use (L1), (L2), (L3) and the existence of a fixpoint formula φ for $\psi(x) := \mathrm{prov}_\Phi(x) \to \eta$ to show that

$$\Phi \vdash \mathrm{prov}_\Phi(\underline{\ulcorner \eta \urcorner}) \to \eta \quad \Rightarrow \quad \Phi \vdash \eta.$$

Hint: from $\Phi \vdash \varphi \to \big(\mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \to \eta\big)$ obtain, by applications of (L1) and (L2), that

$$\Phi \vdash \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \to \big(\mathrm{prov}_\Phi(\underline{\ulcorner \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \urcorner}) \to \mathrm{prov}_\Phi(\underline{\ulcorner \eta \urcorner})\big).$$

Using the assumption on η and (L1)–(L3), one obtains that $\Phi \vdash \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \to \eta$, that $\Phi \vdash \varphi$, $\Phi \vdash \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner})$, and finally $\Phi \vdash \eta$.