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Exercises No.11

Exercise 1 [warm-up no. 1] An FO-definable arithmetical encoding of (fixed length) tuples of natural numbers can be based on Cantor's diagonal enumeration. Consider the bijective function $D\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, where $D(k, \ell) = n$ if (k, ℓ) is the n-th element in the sequence

$$\underbrace{(0,0)}_{\Sigma=0}, \underbrace{(0,1),(1,0)}_{\Sigma=1}, \underbrace{(0,2),(1,1),(2,0)}_{\Sigma=2}, \underbrace{(0,3),(1,2),(2,1),(3,0)}_{\Sigma=3}, \ldots$$

obtained by listing pairs (k, ℓ) in increasing order w.r.t. their sum $\Sigma = k + \ell$ and w.r.t. increasing first component k within each such block. [Think of traversing the nodes of the $\mathbb{N} \times \mathbb{N}$ grid in a succession of northwest-to-southeast diagonal rows.]

Give an explicit definition of $D(k, \ell)$ as a function in k and ℓ. Use this to provide representations of D and its inverse $(p_1, p_2)\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ (defined by the requirement that $D(p_1(n), p_2(n)) = n$) by arithmetical formulae in $\mathfrak{N}$. E.g., for D, a formula $\varphi(x, y, z) \in \mathrm{FO}(\sigma_{\mathrm{ar}})$ such that $\mathfrak{N} \models \varphi[k, \ell, n]$ iff $n = D(k, \ell)$.

Exercise 2 [warm-up no. 2] Provide a formula $\chi(u, v, x, y)$ in $\mathrm{FO}(\sigma_{\mathrm{ar}})$ that defines (represents) Gödel's β-function $\beta(a, b, i) = a \bmod (1 + (i + 1)b)$ in $\mathfrak{N}$:

$$\mathfrak{N} \models \chi[a, b, c, d] \quad \text{iff} \quad d = \beta(a, b, c).$$

Exercise 3 Use the definability of Gödel's β-function in $\mathfrak{N}$ to provide FO-definitions (representations) of one of the following functions over $\mathfrak{N}$:

(a) exponentiation: $(n, m) \mapsto n^m$ as a function from $\mathbb{N}^2$ to $\mathbb{N}$.

(b) factorial: $n \mapsto n!$ as a function from $\mathbb{N}$ to $\mathbb{N}$.

E.g., for (a) one wants to have a formula $\varphi(x, y, z) \in \mathrm{FO}(\sigma_{\mathrm{ar}})$ such that for all $n, m, k \in \mathbb{N}$: $\mathfrak{N} \models \varphi[n, m, k]$ iff $k = n^m$.

Exercise 4 If we think of the 'halting' of a run of an R-program as eventually producing an infinite repetition of the halting configuration, then every run generates an infinite sequence of configurations that may either be

– eventually periodic (if the computation halts or goes into some non-trivial cycle), or

– eventually aperiodic (if the computation neither halts nor enters a cycle).

Define AP in analogy with the halting set H as the set of codes of those R-programs that display eventually aperiodic behaviour on empty input. Show that H and AP are *recursively inseparable* in the sense that there is no decidable set that contains H and is disjoint from AP. Hint: think of a reduction that modifies programs so that they either halt or display eventually aperiodic behaviour.

Exercise 5 Consider the complete FO-theory $\mathrm{Th}(\mathfrak{Z})$ of integer arithmetic,

$$\mathfrak{Z} = (\mathbb{Z}, +^{\mathfrak{Z}}, \cdot^{\mathfrak{Z}}, 0^{\mathfrak{Z}}, 1^{\mathfrak{Z}}).$$

Show that $\mathrm{Th}(\mathfrak{Z})$ is undecidable, via a reduction of $\mathrm{Th}(\mathfrak{N})$ to $\mathrm{Th}(\mathfrak{Z})$, based on the fact that $\mathbb{N}$ is FO-definable as a subset of $\mathbb{Z}$ in $\mathfrak{Z}$. By the "four squares theorem" of elementary number theory, for all $z \in \mathbb{Z}$: $z \in \mathbb{N}$ iff there are $a, b, c, d \in \mathbb{Z}$ such that $z = a^2 + b^2 + c^2 + d^2$.

Find a translation $\varphi \mapsto \varphi^*$ such that $\varphi \in \mathrm{Th}(\mathfrak{N})$ iff $\varphi^* \in \mathrm{Th}(\mathfrak{Z})$ and conclude that $\mathrm{Th}(\mathfrak{Z})$ is undecidable. The translation can be given by induction on syntax, such that, if $\varphi = \varphi(\bar{x})$ then $\varphi^* = \varphi^*(\bar{x})$ (same free variables) and for all $\bar{n}$ over $\mathbb{N}$: $\mathfrak{N} \models \varphi[\bar{n}]$ iff $\mathfrak{Z} \models \varphi^*[\bar{n}]$.