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Exercises No. 10

Exercise 1 Let $\Phi \subseteq \mathrm{FO}(\sigma)$ be recursively enumerable as $\Phi = \{\varphi_n \colon n \in \mathbb{N}\}$ for some total recursive function $n \mapsto \varphi_n$. Argue that then $\hat{\Phi} := \{\psi_n \colon n \in \mathbb{N}\}$ where $\psi_n = \bigwedge_{i=0}^{n} \varphi_i$ is recursive and that $\Phi^{\vdash} = \hat{\Phi}^{\vdash}$. (It follows that any theory that admits a recursively enumerable axion system also has a recursive axiom system.)

Exercise 2 Fill in the details in the proof of Theorem 4.2.2. (Church/Turing). Analogously to the proof of Theorem 4.2.1 (Trakhtenbrot), argue on the basis of our observations concerning φ_{P} that

$$F^* \colon \ulcorner \mathrm{P} \urcorner \longmapsto \varphi_{\mathrm{P}}^*$$

is a reduction from $\bar{\mathrm{H}}$ to SAT(FO) showing that SAT(FO) is not decidable (not r.e.).

Exercises No. X: Some Specials

The following holiday exercises are meant as challenges across several themes we have touched on so far. Some should be manageable, others may be more open-ended ...

Exercise 3

(a) Argue that the following settings do not support a compactness theorem as we know it from FO (i.e., that any set of sentences such that every finite subset is satisfiable is satisfiable as a whole):
  (i) FO w.r.t. satisfiability in *finite models*.
  (ii) $\mathrm{FO}^{<}(\{\sigma\} \dot{\cup} \{<\})$ w.r.t. satisfiability in countable models that interpret the binary relation $<$, which is not in σ, as a linear ordering of the isomorphism type of $(\mathbb{N}, <)$.
  (iii) Monadic second-order logic MSO, which allows quantification over subsets of the universe.

(b) Try to devise compactness-based arguments to show that the following are not definable by single FO-sentences even in restriction to just *finite* structures:
  (i) for a linear ordering $<$, that its length is even.
  (ii) connectivity (of finite graphs).

  NB: Both are definable in MSO in restriction to finite structures.

Exercise 4 Show that satisfiability of the class of $\exists^*\forall^*$-formulae without function symbols in FO is decidable. An FO-formula is an $\exists^*\forall^*$-formula if it consists of a string of existential quantifiers, followed by a string of universal quantifiers and a quantifier-free kernel formula. Hint: it may help to reason that satisfiable $\exists^*\forall^*$-formulae possess finite models.

Exercise 5 First-order logic is *closed under relativisation*: for every first-order formula to $\varphi(\mathbf{x}) \in \mathrm{FO}(\sigma)$ (in a relational signature σ) and unary relation symbol $U \notin \sigma$, there is a formula $\varphi^U(\mathbf{x}) \in \mathrm{FO}(\sigma \mathbin{\dot\cup} \{U\})$ such that, for every σ-structure $\mathfrak{A}$ and $U^{\mathfrak{A}} \subseteq A$, and for every tuple $\mathbf{a}$ in $U^{\mathfrak{A}}$,

$$(\mathfrak{A}, U^{\mathfrak{A}}) \models \varphi^U[\mathbf{a}] \quad \text{iff} \quad \mathfrak{A}\restriction U^{\mathfrak{A}} \models \varphi[\mathbf{a}].$$

Define a map $\varphi \mapsto \varphi^U$ by syntactic induction and show that it satisfies the requirements.

Exercise 6 Sketch in outline an argument to show that first-order graph theory

$$\big\{\varphi \in \mathrm{FO}_0(\{E\}) \colon (A, E^{\mathfrak{A}}) \models \varphi \text{ for all graphs } (A, E^{\mathfrak{A}})\big\}$$

is undecidable. Is it r.e.? How about the correspondingly defined theory of all *finite* graphs?

Exercise 7 Assuming that ZFC has a model, it must also have

(i) a countable model,

(ii) non-standard models in which, e.g., the order type of $(\mathbb{Z}, <)$ can be embedded into the ordering of ω by $\in$.

Discuss how these phenomena *are* compatible with the fact that ZFC implies that

(i) the powerset of ω is uncountable,

(ii) the principle of induction is valid for $(\omega, \in)$, i.e., the internal successor structure on ω satisfies the Peano axioms.

Exercise 8 A *tiling system* $\mathfrak{D} = (D, H^{\mathfrak{D}}, V^{\mathfrak{D}})$ consists of a finite set D of tile types with two binary relations H, V. That $(d, d') \in H^{\mathfrak{D}}$ means that tile d' may be put, as a right-hand neighbour, horizontally next to tile d, and similarly for $V^{\mathfrak{D}}$ and vertical neighbours.

A tiling of a quadaratic grid structure $\mathfrak{G} = (G, H^{\mathfrak{G}}, V^{\mathfrak{G}})$ with vertex set G and horizontal and vertical successor relations $H^{\mathfrak{G}}$ and $V^{\mathfrak{G}}$ is a homomorphism $h \colon \mathfrak{G} \to \mathfrak{D}$. We look at tilings of the natural grids $\mathfrak{G}_{\mathbb{N}}$ on $\mathbb{N} \times \mathbb{N}$, $\mathfrak{G}_{\mathbb{Z}}$ on $\mathbb{Z} \times \mathbb{Z}$ and finite tori $\mathfrak{G}_{n,m}$ of periods $n, m \geqslant 2$ obtained from these.

The tiling problems T and T_0 are defined as decision problems for the sets T of (encodings of) those $\mathfrak{D}$ that admit a tiling of $\mathfrak{G}_{\mathbb{Z}}$, and T_0 of those $\mathfrak{D}$ that admit a tiling of some $\mathfrak{G}_{n,m}$.

Clearly $\mathrm{T}_0 \subseteq \mathrm{T}$ (why?). Both problems are undecidable, and it is even known that there is no decidable set that contains T_0 and is contained in T (recursive inseparability).

(a) Show that $\mathfrak{D}$ tiles $\mathfrak{G}_{\mathbb{N}}$ iff $\mathfrak{D}$ tiles $\mathfrak{G}_{\mathbb{Z}}$ iff $\mathfrak{D}$ tiles arbitrarily large finite square grids.

(b) Try to devise (in outline) a simultaneous reduction of T_0 and T to the finite and general satisfiability problem for FO, confirming that FINSAT(FO) and SAT(FO) are recursively inseparable, too.