Introduction to Mathematical Logic

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Exercises No.1

Exercise 1 Describe *all* substructures of each of the following structures:

(1) $\mathfrak{A}_1 := (\mathbb{N}, \mathsf{S}, 0)$ in signature $\sigma = \{f, c\}$ with a unary function symbol f and a constant symbol c, where f is interpreted as the *successor function* $\mathsf{S}\colon n \mapsto n+1$.

(2) $\mathfrak{A}_2 := (\mathbb{N}, \mathsf{S})$, the reduct of the structure in (1) to $\sigma' = \{\mathsf{S}\}$.

(3) $\mathfrak{A}_3 := (\mathbb{N}, \mathrm{graph}(\mathsf{S}))$, with the graph of the successor function S as the interpretation of a binary relation symbol.

(4) $\mathfrak{A}_4 := \mathfrak{N} = (\mathbb{N}, +^{\mathfrak{N}}, \cdot^{\mathfrak{N}}, <^{\mathfrak{N}}, 0, 1)$ in signature σ_{ar} with the standard interpretation.

(5) $\mathfrak{A}_5 := (\mathbb{N}, +^{\mathfrak{N}}, <^{\mathfrak{N}}, 0)$ as a reduct of the structure in (4).

NB: In some cases, distinct but isomorphic substructures occur.

Exercise 2 Show that for any signature σ and σ-structure $\mathfrak{A}$, the set of *automorphisms* of $\mathfrak{A}$, i.e., the set of all isomorphisms $f\colon \mathfrak{A} \simeq \mathfrak{A}$, forms a group with the binary operation of composition and with the identity map on A as the neutral element. This is called the *automorphism group* of $\mathfrak{A}$, $\mathrm{Aut}(\mathfrak{A})$.

Exercise 3 Which of the following classes of σ-structures are closed under taking homomorphic images, and under passage to substructures, respectively?

(i) Groups in the signature $\sigma = \{\circ, e\}$.

(ii) Groups in the signature $\sigma = \{\circ, e, {}^{-1}\}$.

(iii) Fields and 1-element rings in the signature $\sigma = \{+, \cdot, 0, 1\}$.

(iv) Strict linear orderings in the signature $\sigma = \{<\}$.

(v) Simple undirected graphs in the signature $\sigma = \{E\}$.

Exercise 4 Recall that a set is *countable* if it is finite or bijectively related to the set of natural numbers. Recall also that unions of countably many countable sets, as well as finite products of countable sets, are again countable. A structure is called countable if its universe is a countable set. Show the following, for any countable signature σ:

(a) T_σ and $\mathrm{FO}(\sigma)$ are countable.

(b) For any σ-structure $\mathfrak{A}$ with universe A and any countable subset $A_0 \subseteq A$ there is a countable substructure $\mathfrak{B} \subseteq \mathfrak{A}$ with universe B such that $A_0 \subseteq B \subseteq A$.

Exercise 5 Let σ be a signature consisting of only function and constant symbols, and with $\mathrm{const}(\sigma) \neq \emptyset$. Let $\mathfrak{T}_\sigma$ be the *free or term σ-structure* with universe T_σ. With a subset $V_0 \subseteq \mathrm{Var}$ associate the set of terms $T_\sigma(V_0) := \{t \in T_\sigma : \mathrm{var}(t) \subseteq V_0\}$.

(a) Show that $T_\sigma(V_0)$ is the universe of a substructure $\mathfrak{T}_\sigma(V_0) := \mathfrak{T}_\sigma \restriction T_\sigma(V_0) \subseteq \mathfrak{T}_\sigma$.

(b) Show that for any σ-structure $\mathfrak{A}$ there is a unique homomorphism $h\colon \mathfrak{T}_\sigma(\emptyset) \xrightarrow{\mathrm{hom}} \mathfrak{A}$.

(c) Let $\beta_0\colon V_0 \to A$ be a *partial assignment* in the σ-structure $\mathfrak{A}$. Show that there is a unique homomorphism $h\colon \mathfrak{T}_\sigma(V_0) \xrightarrow{\mathrm{hom}} \mathfrak{A}$ that extends β_0 [in fact: the restriction of the interpretation function of terms for any assignment that extends β_0].

Exercise 6 [extra]
Consider relational structures $\mathfrak{A} = (A, R^{\mathfrak{A}})$ with a relation R of arity r.

(a) The structure $\mathfrak{A}_0 = (A_0, R^{\mathfrak{A}_0})$ is a *weak substructure* of $\mathfrak{A}$, $\mathfrak{A}_0 \subseteq_{\mathrm{w}} \mathfrak{A}$, if $A_0 \subseteq A$ and $R^{\mathfrak{A}_0} \subseteq R^{\mathfrak{A}}$. Show that homomorphic images are weak substructures of the target structure.

(b) A weak substructure $\mathfrak{A}_0 \subseteq_{\mathrm{w}} \mathfrak{A}$ is called a *core* if it is a $\subseteq_{\mathrm{w}}$-minimal homomorphic image of $\mathfrak{A}$ within $\mathfrak{A}$: there is a homomorphism $h\colon \mathfrak{A} \to \mathfrak{A}$ s.t. $\mathfrak{A}_0 = h(\mathfrak{A})$ and if $h'\colon \mathfrak{A} \to \mathfrak{A}$ is any homomorphism with $h'(\mathfrak{A}) \subseteq_{\mathrm{w}} \mathfrak{A}_0$ then $h'(\mathfrak{A}) = \mathfrak{A}_0$. Show the following for finite $\mathfrak{A}$.

   (i) $\mathfrak{A}$ has a core.

   (ii) All cores of $\mathfrak{A}$ are pairwise isomorphic.

   (iii) Every core of $\mathfrak{A}$ is a *retract* of $\mathfrak{A}$, i.e., a weak substructure $\mathfrak{A}_0 \subseteq_{\mathrm{w}} \mathfrak{A}$ that is the image of $\mathfrak{A}$ under some homomorphism $h\colon \mathfrak{A} \mapsto \mathfrak{A}$ that fixes A_0 pointwise.

(c) $\mathfrak{A}$ and $\mathfrak{B}$ are called *homomorphically equivalent* if there are homomorphisms $h_1\colon \mathfrak{A} \to \mathfrak{B}$ and $h_2\colon \mathfrak{B} \to \mathfrak{A}$. Show that two finite R-structures are homomorphically equivalent if, and only if, their cores are isomorphic.