1 Recap on Recursive Analysis

1.1 Notions of computability for real numbers

Definition 1.1. *a) Call $x \in \mathbb{R}$ binarily computable iff there exists a computable sequence $b_n \in \{0,1\}$, $n \geq -N$, (i.e. a function $b : \{-N, -N+1, \ldots, 0, 1, 2, \ldots\} \to \{0,1\}$) such that $\sum_{n=-N}^{\infty} b_n 2^{-n}$.*

b) Call $x \in \mathbb{R}$ computable iff there exists a computable integer sequence $(c_n)_n$ such that $|x - c_n/2^{n+1}| \leq 2^{-n}$.

c) Let $\mathbb{D}_n := \{c/2^n \mid c \in \mathbb{Z}\}$ and $\mathbb{D} := \bigcup_n \mathbb{D}_n$ denote the set of dyadic rationals.

d) Call $x \in \mathbb{R}$ Cauchy-computable iff there exist computable sequences $q_n, \varepsilon_n \in \mathbb{Q}$ such that $|x - q_n| \leq \varepsilon_n \to 0$ as $n \to \infty$.

e) Call $x \in \mathbb{R}$ naively computable iff there exists a computable sequence $q_n \in \mathbb{Q}$ such that $q_n \to x$ as $n \to \infty$.

f) A sequence $s_n \in \{\mathtt{1}, \mathtt{0}, \bar{\mathtt{1}}\}$, $n \geq -N$, is called a signed digit expansion of $\sum_{n=-N}^{\infty} s_n 2^{-n}$.
Encoded over $\{0,1\}^\omega$, $\big(\mathrm{bin}(N), (s_n)_n\big)$ is a ρ_{sd}–name of x.
Call a sequence $(b_n)_n$ as in a) (encoded over $\{0,1\}^\omega$) a ρ_b–name of x;
and $(c_n)_n$ as in b) a ρ–name of x.
A pair $(q_n)_n$ and $(\varepsilon_n)_n$ of sequences as in d) is a ρ_C–name of x.
A sequence $(q_n)_n$ as in e) is a ρ_n–name of x.

Lemma 1.2. *a) Every binarily computable real has a computable signed digit expansion.*

b) Every real with a computable signed digit expansion is computable.

c) Every computable real is Cauchy-computable d) and vice versa.

e) (Cauchy-)computability implies naive computability,

Example 1.3 *a) Every rational number $x \in \mathbb{Q}$ is binarily computable.*

b) $\sqrt{2}$ and π are (Cauchy-)computable real numbers.

c) For $H \subseteq \mathbb{N}$ the Halting problem, $\sum_{n \in H} 2^{-n}$ is not binarily computable

d) but naively computable.

1.2 Computing functions and relations on a continuous universe

Definition 1.4. *a) A multivalued (possibly partial) function $f :\subseteq X \rightrightarrows Y$ (aka relation or multifunction) is a subset of $X \times Y$.*
We write $\mathrm{dom}(f) := \{x \in X \mid \exists y \in Y : (x,y) \in f\}$ and $f(x) = \{y \in Y \mid (x,y) \in f\}$.

b) A Type-2 Machine has an infinite read-only input tape, an infinite one-way output tape, and an unbounded work tape.
It computes a (possibly partial) function $F :\subseteq \{0,1\}^\omega \to \{0,1\}^\omega$.

c) A representation of a set X is a partial surjective mapping $\alpha :\subseteq \{0,1\}^\omega \to X$.

d) Fix representations α of X and β of Y and a (possibly partial and multivalued) function $f :\subseteq X \rightrightarrows Y$.
A $(\alpha \to \beta)$–realizer of f is a (partial but single-valued) function $F :\subseteq \{0,1\}^\omega \to \{0,1\}^\omega$ with $f\big(\alpha(\bar\sigma)\big) \ni \beta\big(F(\bar\sigma)\big)$ for every $\bar\sigma \in \mathrm{dom}(F) := \{\bar\sigma \mid \alpha(\bar\sigma) \in \mathrm{dom}(f)\}$.

e) *A function as in d) is* $(\alpha \to \beta)$*–computable if it has a computable realizer in the sense of b). (We simply say computable if* α, β *are clear from context.) It is* $(\alpha \to \beta)$*–continuous if it has a continuous realizer.*

f) *Let* α_i *be representations for* X_i, $i \in I \subseteq \mathbb{N}$, *and* $\langle \cdot \mid \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *a computable surjective pairing function. Then* $(\sigma_m)_m$ *is a* $\left(\prod_{i\in I} \alpha_i\right)$*–name of* $(x_i)_i \in \prod_i X_i$ *iff* $(\sigma_{\langle i,n\rangle})_n$ *is an* α_i*–name of* $x_i \in X_i$ *for every* $i \in I$.

Example 1.5 *a) Let* α, β, γ *denote representations of* X, Y, Z*, respectively. If* $f :\subseteq X \rightrightarrows Y$ *is* $(\alpha \to \beta)$*–computable and* $g :\subseteq Y \rightrightarrows Z$ *is* $(\beta \to \gamma)$*–computable, then so is their composition*

$$g \circ f \;:=\; \left\{(x,z) \,\middle|\, x \in X, z \in Z, f(x) \subseteq \mathrm{dom}(g),\, \exists y \in Y : (x,y) \in f \wedge (y,z) \in g\right\} \;. \tag{1}$$

b) *A single-valued total real function* $f : [0,1] \to \mathbb{R}$ *is* $(\rho \to \rho)$*–computable if some Type-2 machine can map every* ρ*–name* $(c_n)_n$ *of some* $x \in [0,1]$ *to a* ρ*–name* $(c'_m)_m$ *of* $f(y)$.

c) *Addition and multiplication are* $(\rho \times \rho \to \rho)$*–computable;*
inversion $\mathbb{R} \setminus \{0\} \ni x \mapsto 1/x$ *is* $(\rho \to \rho)$*–computable.*

d) *Every polynomial with computable coefficients is computable; and vice versa.*

e) *Let* $(a_n)_n$ *denote a computable sequence,* $R := 1/\limsup_n \sqrt[n]{|a_n|}$ *and* $0 < r < R$. *Then* $[-r,r] \ni x \mapsto \sum_n a_n x^n$ *is computable. In particular* $\exp, \sin, \cos, \ln(1+x)$ *are computable.*

f) *Fix* $\varepsilon > 0$. *The multifunction* $\widetilde{\mathrm{sgn}}_\varepsilon : \mathbb{R} \rightrightarrows \{-1,+1\}$ *with* $\varepsilon > x \mapsto -1$ *and* $-\varepsilon < x \mapsto +1$ *is computable.*

g) *Any* $x \in \mathbb{R}$ *is binarily computable iff it is computable.*

Theorem 1.6. *a) Every (oracle-)computable* $F :\subseteq \{0,1\}^\omega \to \{0,1\}^\omega$ *is continuous.*

b) *To every continuous* $F :\subseteq \{0,1\}^\omega \to \{0,1\}^\omega$*, there exists an oracle relative to which* F *becomes computable.*

c) *Every oracle-computable* $f : [0,1] \to \mathbb{R}$ *is continuous!*

d) *There exists a computable sequence of (degrees and coefficient lists of) univariate dyadic polynomials* $P_n \in \mathbb{D}[X]$ *with* $\big\|P_n(x) - |x|\big\| \le 2^{-n}$ *on* $[-1,+1]$.

e) *Fix an oracle* $\mathcal{O}$. *Continuous (total)* $f : [0,1] \to \mathbb{R}$ *is computable relative to* $\mathcal{O}$ *iff there exists a sequence* $P_n \in \mathbb{D}_{n+1}[X]$ *computable relative to* $\mathcal{O}$ *such that* $\|f - P_n\|_\infty \le 2^{-n}$.

f) *To every continuous* $f : \mathbb{R} \to \mathbb{R}$ *there is an oracle relative to which* f *becomes computable.*

1.3 Encoding functions and closed subsets

Definition 1.7. *a) A* $[\rho^d \to \rho]$*–name of* $f \in C(\mathbb{R}^d)$ *is a double sequence* $P_{n,m} \in \mathbb{D}[X_1, \ldots, X_d]$ *with* $|f(x) - P_{n,m}(x)| \le 2^{-n}$ *for all* $\|x\| \le m$.

b) *A closed set* $A \subseteq \mathbb{R}^d$ *is* computable *if the function*

$$\mathrm{dist}_A : \mathbb{R}^d \;\ni\; x \;\mapsto\; \min\left\{\|x-a\| : a \in A\right\} \;\in\; \mathbb{R} \cup \{\infty\} \;=:\; \overline{\mathbb{R}} \tag{2}$$

is computable. A ψ^d*–name of* $A \in \mathcal{A}^{(d)}$ *is a* $[\rho^d \to \rho]$*–name of* dist_A*, where* $\mathcal{A}^{(d)}$ *denotes the space of closed subsets of* $\mathbb{R}^d$.

c) *A* $\psi^d_<$*–name of* A *is a* $\left(\prod_{m\in\mathbb{N}} \rho^d\right)$*–name of some sequence* $x_m \in A$ *dense in* A.

d) A $\psi^d_>$–name of A are two sequences $q_n \in \mathbb{Q}^d$ and $\varepsilon_n \in \mathbb{Q}$ such that

$$\mathbb{R}^d \setminus A \;=\; \bigcup_n B(q_n,\varepsilon_n) \quad \textit{where} \quad B(x,r) := \{y : \|x-y\| < r\} \ . \tag{3}$$

*e) A $\rho_<$–name of $x \in \mathbb{R}$ is a sequence $(q_n) \subseteq \mathbb{Q}$ with $x = \sup_n q_n$;
a $\rho_>$–name of $x \in \mathbb{R}$ is a sequence $(q_n) \subseteq \mathbb{Q}$ with $x = \inf_n q_n$.*

f) For representations α, β of X let $\alpha \sqcap \beta := (\alpha \times \beta)\big|^{\Delta_X}$, where $\Delta_X := \{(x,x) \mid x \in X\}$.

g) Write $\alpha \preceq \beta$ if $\mathrm{id} : X \to X$ is $(\alpha \to \beta)$–computable.

h) We say that $U \subseteq X$ is α–r.e. if there exists a Turing machine which terminates precisely on input of all α–names of $x \in U$ and diverges on all α–names of $x \in X \setminus U$.

Theorem 1.8. *a) It holds $\rho \preceq \rho_< \sqcap \rho_> \preceq \rho$.*

b) Every $(\rho \to \rho_<)$–computable $f : [0,1] \to \mathbb{R}$ is lower semi-continuous.

c) A set $A \in \mathcal{A}^{(d)}$ is $\psi^d_>$–computable iff $\mathbb{R}^d \setminus A$ is ρ^d–r.e.

d) Let $\|\cdot\|$ in Equation 3 denote any fixed computable norm. Let $\|\cdot\|'$ denote some other norm and $\psi'^d_>$ the induced representation. Then $\psi^d_> \preceq \psi'^d_>$.

*e) It holds $\psi^d \preceq \psi^d_< \sqcap \psi^d_> \preceq \psi^d$.
Moreover A is $\psi^d_<$–computable iff dist_A is $(\rho^d \to \rho_>)$–computable;
and A is $\psi^d_>$–computable iff dist_A is $(\rho^d \to \rho_<)$–computable.
In particular ψ^d–computability is invariant under a change of computable norms.*

*f) Union $\mathcal{A}^{(d)} \times \mathcal{A}^{(d)} \ni (A,B) \mapsto A \cup B \in \mathcal{A}^{(d)}$ is $(\psi^d \times \psi^d \to \psi^d)$–computable;
but intersection is not.*

g) Closed image $C(\mathbb{R}^d \to \mathbb{R}^k) \times \mathcal{A}^{(d)} \ni (f,A) \mapsto \overline{f[A]} \in \mathcal{A}^{(k)}$ is $([\rho^d \to \rho^k] \times \psi^d_<, \psi^k_<)$–computable.

h) Preimage $C(\mathbb{R}^d \to \mathbb{R}^k) \times \mathcal{A}^{(k)} \ni (f,B) \mapsto f^{-1}[B] \in \mathcal{A}^{(d)}$ is $([\rho^d \to \rho^k] \times \psi^k_>, \psi^d_>)$–computable.

j) $\{\emptyset\}$ is $\psi^d_>\big|^{[0,1]^d}$–r.e.

2 (In-)Computability in Linear Algebra and Geometry

Common algorithms (e.g. Gaussian Elimination) generally pertain to the Blum-Shub-Smale model (equivalently: *real-RAM*) of real computation — and lead to difficulties when implemented.

Definition 2.1. *a) For a set $S \subseteq \mathbb{R}^d$, its* convex hull *is the least convex set containing S:*

$$\mathrm{chull}(S) \;:=\; \bigcap \big\{C : S \subseteq C \subseteq \mathbb{R}^d, C \textit{ convex}\big\} \ .$$

A polytope *is the convex hull of finitely many points,* $\mathrm{chull}(\{p_1,\ldots,p_N\})$. *For a convex set C, point $p \in C$ is called* extreme *(written "$p \in \mathrm{ext}(C)$") if it does not lie on the interior of any line segment contained in C:*

$$p = \lambda \cdot x + (1-\lambda)\cdot y \ \wedge\ x,y \in C \ \wedge\ 0 < \lambda < 1 \quad \Rightarrow \quad x = y \ .$$

b) *For a set X, let $\binom{X}{k} := \big\{\{x_1,\dots,x_k\} : x_i \in X$ pairwise distinct$\big\}$.* Convex Hull, *as understood in computational geometry, is the problem*

$$\mathsf{extchull}_N : \binom{\mathbb{R}^d}{N} \ni \{x_1,\dots,x_N\} \mapsto \big\{y \text{ extreme point of } \mathrm{chull}(x_1,\dots,x_N)\big\} \tag{4}$$

of identifying the extreme points of the polytope C spanned by given pairwise distinct $x_1,\dots,x_N$.

c) *For $1 \le j \le n$ let $g_j : \mathbb{R}^d \ni x \mapsto a_{j0} + \sum_i x_i \cdot a_{ji} \in \mathbb{R}$ ($(a_{j1},\dots,a_{jd}) \neq 0$) denote an affine linear function and $H_j = g_j^{-1}(0)$ its induced oriented hyperplane, $H_j^+ = g_j^{-1}(0,\infty)$ the positive halfspace and $H_j^- = g_j^{-1}(-\infty,0)$ the negative one. For $x \in \mathbb{R}^d$,*

$$\pi(x) \quad = \quad \big(\operatorname{sgn} g_j(x)\big)_{j=1,\dots,n} \quad \in \quad \{-1,0,+1\}^n$$

is called its position vector. *A* cell *of $\mathcal{H} = \{H_1,\dots,H_n\}$ is a subset $\pi^{-1}(\sigma)$ of $\mathbb{R}^d$ for some $\sigma \in \{-1,+1\}^n$.* Point Location *is the problem of identifying, to fixed $\mathcal{H}$ and upon input of some point x, the cell that x lies in, i.e., of computing the function*

$$\mathsf{Pointloc}_{\mathcal{H}} : \mathbb{R}^d \setminus \bigcup \mathcal{H} \to \{-1,+1\}^n, \qquad x \mapsto \pi(x) \ . \tag{5}$$

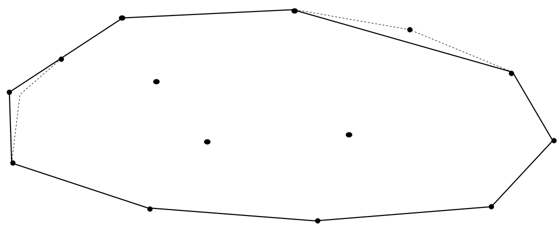
d) *Let $\det_d : \mathbb{R}^{d\times d} \to \mathbb{R}$ denote the determinant mapping and $\mathrm{rank}_{d,k} : \mathbb{R}^{d\times k} \to \mathbb{N}$ the rank. Moreover abbreviate $\mathrm{Gr}_k(V) := \big\{U \subseteq V$ lin.subspace, $\dim(U) = k\big\}$ and $\mathrm{Gr}(V) = \bigcup_k \mathrm{Gr}_k(V)$. Finally,* $\mathsf{GramSchmidt}_{d,k} : \mathrm{rank}_{d,k}^{-1}[k] \to \mathbb{R}^{d\times k}$ *is the mapping induced by the Gram–Schmidt orthonormalization process.*

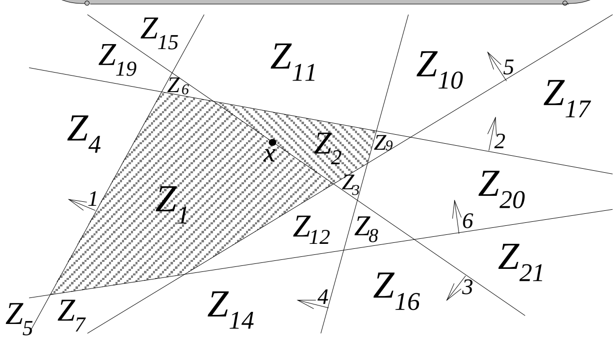
e) *Consider the multifunctions*

$$\mathsf{LSolve}_{d,k} : \mathrm{rank}_{d,k}^{-1}[\{0,1,\dots,k-1\}] \ni A \rightrightarrows \ker(A) \setminus \{0\}$$

and $\mathsf{LSolve}'_{d,k} :\subseteq \mathbb{R}^{d\times(k+1)} \ni (A,b) \rightrightarrows \big\{x \mid A\cdot x = b\big\}$.

f) *Consider the multifunctions* $\mathsf{SomeEVec}_d : \mathbb{R}^{d\times(d+1)/2} \ni A \rightrightarrows \big\{x \neq 0 \mid \exists\lambda : A\cdot x = \lambda x\big\}$ *and* $\mathsf{EVecBase}_d : \mathbb{R}^{d\times(d+1)/2} \ni A \rightrightarrows \big\{O \in \mathcal{O}(\mathbb{R}^d) \mid O^\dagger \cdot A \cdot O = \mathrm{diag}(\cdots)\big\}$.





Proposition 2.2. *a)* $\mathsf{extchull}_N$ *is discontinuous, hence incomputable;*

b) For any $\mathcal{H} \neq \emptyset$, $\mathsf{Pointloc}_{\mathcal{H}}$ *is discontinuous, hence incomputable;*

c) det_d *and* $\mathsf{GramSchmidt}_{d,k}$ *are computable;*

d) $\mathrm{rank}_{d,k}$ *is discontinuous (hence incomputable) but* $(\rho^{d\times k}, \rho_<)$*–computable.*

e) Linear independence $\big\{(x_1,\ldots,x_k) \in \mathbb{R}^{d\times k}$ *linearly independent*$\big\}$ *is* $\rho^{d\times k}$*–r.e.*

f) $\mathsf{LSolve}_{d,k}$ *and* $\mathsf{SomeEVec}_d$ *are uncomputable.*

Theorem 2.3. *a)* $\dim_d : \mathrm{Gr}(\mathbb{R}^d) \to \{0,1,\ldots,d\}$ *is* $(\psi^d_<, \rho_<)$*–computable*

b) and $(\psi^d_>, \rho_>)$*–computable; that is,* (ψ^d, ρ)*–computable!*

Lemma 2.4. *a) The generalized determinant is* $(\rho^{d\times k}, \rho)$*–computable, namely the mapping* $\mathrm{Det}_{d\times k} : \mathbb{R}^{d\times k} \ni (a_1,\ldots,a_k) \mapsto \max\big\{|\det(a_{j_1},\ldots,a_{j_d})| : 1 \leq j_1 \leq \cdots \leq j_d \leq k\big\}$.

b) $\mathbb{R}^{d\times k} \ni A \mapsto \mathrm{range}(A) \in \mathcal{A}^{(d)}$ *is* $(\rho^{d\times k}, \psi^d_<)$*–computable.*

c) $\mathbb{R}^{d\times k} \ni A \mapsto \mathrm{kern}(A) \in \mathcal{A}^{(k)}$ *is* $(\rho^{d\times k}, \psi^k_>)$*–computable.*

d) $\mathbb{R}^{d\times k} \cap \mathrm{rank}^{-1}[k] \ni A \mapsto \mathrm{range}(A) \in \mathcal{A}^{(d)}$ *is* $(\rho^{d\times k}, \psi^d_>)$*–computable.*

e) Orthogonal complement, i.e. the mapping $\mathrm{Gr}(\mathbb{R}^d) \ni L \mapsto L^\perp \in \mathrm{Gr}(\mathbb{R}^d)$*, is both* $(\psi^d_<, \psi^d_>)$*–computable and* $(\psi^d_>, \psi^d_<)$*–computable.*

f) The multivalued mapping $\mathrm{Basis}_{d,k} : \mathrm{Gr}_k(\mathbb{R}^d) \ni L \rightrightarrows \big\{B \in \mathbb{R}^{d\times k} : \mathrm{range}(B) = L\big\}$ *is* $(\psi^d_<, \rho^{d\times k})$*–computable.*

Theorem 2.5. *For fixed integers* $0 \leq k \leq d$*, the following representations of* $\mathrm{Gr}_k(\mathbb{R}^d)$ *are uniformly equivalent: A name of* $L \in \mathrm{Gr}_k(\mathbb{R}^d)$ *is*

a) a $\rho^{d\times k}$*-name for some basis* $x_1,\ldots,x_k \in \mathbb{R}^d$ *for L;*

b) same for an orthonormal basis;

c) a $\rho^{d\times m}$*-name (*$m \in \mathbb{N}$ *arbitrary) for some real* $d \times m$*-matrix B with* $L = \mathrm{range}(B)$*;*

d) a $\psi^d_<$*–name of* d_L*, i.e., approximations to* d_L *from above*

e) a $\psi^d_>$*–name of* d_L*, i.e., approximations to* d_L *from below*

f) a $\rho^{m\times d}$*-name (*$m \in \mathbb{N}$ *arbitrary) for some real* $m \times d$*-matrix A with* $L = \mathrm{kern}(A)$*;*

a')–f') similarly, but for $L' := L^\perp$ *and* $k' := d-k$ *instead of L and k.*

Fact 2.6 (E. Specker 1967) *Let* $\mathbb{C}_d[Z]$ *denote the vector space of monic polynomials of degree d. The mapping* $\mathbb{C}^d \ni (z_1,\ldots,z_d) \mapsto \prod_{j=1}^d (Z - z_j) \in \mathbb{C}_d[Z]$ *has a computable multivalued inverse.*

Lemma 2.7. *a) For* $d \in \mathbb{N}$*, given* $x, y_1, \ldots, y_d \in \mathbb{R}$ *and* $\nu := \mathrm{Card}\{1 \leq i \leq d : x = y_i\}$*, one can compute* $(i_1,\ldots,i_\nu)$ *with* $1 \leq i_1 < \cdots < i_\nu \leq d$ *and* $x = y_{i_1} = \cdots = y_{i_\nu}$*.*

b) Given $x_1,\ldots,x_d \in \mathbb{R}$ *and* $k := \mathrm{Card}\{x_1,\ldots,x_d\}$*, one can compute* $\nu_1,\ldots,\nu_d \in \mathbb{N}$ *with* $\nu_j = \mathrm{Card}\{i : 1 \leq i \leq d, x_i = x_j\}$*.*

Theorem 2.8. *a) Given a* $\rho^{d\cdot(d-1)/2}$*–name of a symmetric real* $d\times d$*–matrix* A*,*
a d*–tuple* $(\lambda_1,\ldots,\lambda_d)$ *of its eigenvalues with multiplicities is multivalued* ρ^d*–computable.*

b) Given a $\rho^{d\cdot(d-1)/2}$*–name of a symmetric real* $d\times d$*–matrix* A *and given its number* $\operatorname{Card}\sigma(A)$ *of distinct eigenvalues, one can diagonalize* A *in the sense of* $\rho^{d\times d}$*–computing an orthonormal basis of eigenvectors.*

c) Given a $\rho^{d\cdot(d-1)/2}$*–name of a symmetric real* $d\times d$*–matrix* A *and given the integer*

$$\lfloor \log_2 m\rfloor, \quad \textit{where} \quad m(A) \quad := \quad \min\big\{\dim\ker(A-\lambda\cdot\mathrm{id}) : \lambda\in\sigma(A)\big\} \quad \in \quad \{1,\ldots,d\}$$

denotes the multiplicity of some least-degenerate eigenvalue, one can ρ^d*–compute some eigenvector of* A*.*

Definition 2.9. *For* $1\leq k\leq d$ *integers let* $\mathsf{Class}_{d,k}(x_1,\ldots,x_d) := \big\{j : 1\leq j\leq d, x_j = x_k\big\}$ *and consider the multivalued mapping*

$$\mathsf{SomeClass}_d : \mathbb{R}^d \ni (x_1,\ldots,x_d) \rightrightarrows \big\{\mathsf{Class}_{d,k}(x_1,\ldots,x_d) : 1\leq k\leq d\big\}$$

yielding, for some k*, the set of all indices* i *with* $x_i = x_k$*.*

Lemma 2.10. *Let* $x_1,\ldots,x_d\in\mathbb{R}$ *and* $m := \min_{1\leq k\leq d}\operatorname{Card}\mathsf{Class}_{d,k}(x)$ *as above.*

a) For each $1\leq k,\ell\leq d$ *it either holds* $\mathsf{Class}_{d,\ell}(x) = \mathsf{Class}_{d,k}(x)$ *or* $\mathsf{Class}_{d,\ell}(x)\cap\mathsf{Class}_{d,k}(x) = \emptyset$*. Also,* $\bigcup_k \mathsf{Class}_{d,k}(x) = [d]$*.*

b) Consider $I\subseteq[d]$ *such that*

$$x_i \neq x_j \quad \textit{for all} \quad i\in I \quad \textit{and all} \quad j\in[d]\setminus I \ . \tag{6}$$

Then $I\cap\mathsf{Class}_{d,k}(x)\neq\emptyset$ *implies* $\mathsf{Class}_{d,k}(x)\subseteq I$*.*
Moreover $1\leq\operatorname{Card}(I) < 2m$ *implies* $I = \mathsf{Class}_{d,k}(x)$ *for some* k*.*

c) Suppose $k\in\mathbb{N}$ *is such that* $k\leq m<2k$*.*
Then there exists ℓ *such that* $I := \mathsf{Class}_{d,\ell}(x)$ *satisfies (6) and has* $k\leq\operatorname{Card}(I)<2k$*.*
Conversely every $I\subseteq[d]$ *with* $k\leq\operatorname{Card}(I)<2k$ *satisfying (6) has* $I=\mathsf{Class}_{d,\ell}(x)$ *for some* ℓ*.*

d) Given a ρ^d*–name of* $(x_1,\ldots,x_d)$ *and given* $k\in\mathbb{N}$ *with* $k\leq m<2k$*,*
one can computably find some $\mathsf{Class}_{d,\ell}(x)$*.*

3 Continuity for Multivalued Functions

Definition 3.1. *Let* (X,d) *and* (Y,e) *denote metric spaces and abbreviate* $B(x,r) := \{x'\in X : d(x,x')<r\}\subseteq X$ *and* $\overline{B}(x,r) := \{x'\in X : d(x,x')\leq r\}$*; similarly for* Y*. Now fix some* $f :\subseteq X\rightrightarrows Y$ *and call* $(x,y)\in f$ *a* point of continuity of f *if the following formula holds:*

$$\forall\varepsilon>0\ \exists\delta>0\ \forall x'\in B(x,\delta)\cap\mathrm{dom}(f)\ \exists y'\in B(y,\varepsilon)\cap f(x') \ .$$

a) Call f strongly continuous *if every* $(x,y)\in f$ *is a point of continuity of* f*:*

$$\forall x\in\mathrm{dom}(f)\ \forall y\in f(x)\ \forall\varepsilon>0\ \exists\delta>0\ \forall x'\in B(x,\delta)\cap\mathrm{dom}(f)\ \exists y'\in B(y,\varepsilon)\cap f(x').$$

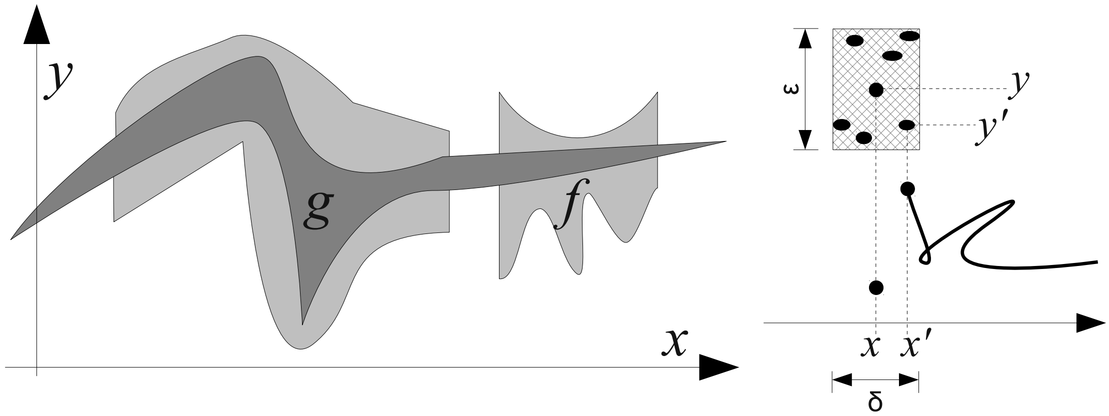


Fig. 1. a) For a relation g (dark gray) to tighten f (light gray) means no more freedom (yet the possibility) to choose some $y \in g(x)$ than to choose some $y \in f(x)$ (whenever possible). b) Illustrating ε–δ–continuity in (x,y) for a relation (black)

b) *Call* f *weakly continuous if the following holds:*

$$\forall x \in \mathrm{dom}(f)\ \exists y \in f(x)\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall x' \in B(x,\delta) \cap \mathrm{dom}(f)\ \exists y' \in B(y,\varepsilon) \cap f(x').$$

c) *Call* f *uniformly weakly continuous if the following holds:*

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathrm{dom}(f)\ \exists y \in f(x)\ \forall x' \in B(x,\delta) \cap \mathrm{dom}(f)\ \exists y' \in B(y,\varepsilon) \cap f(x').$$

d) *Call* f *nonuniformly weakly continuous if the following holds:*

$$\forall \varepsilon > 0\ \forall x \in \mathrm{dom}(f)\ \exists \delta > 0\ \exists y \in f(x)\ \forall x' \in B(x,\delta) \cap \mathrm{dom}(f)\ \exists y' \in B(y,\varepsilon) \cap f(x').$$

e) *Call* f *Henkin-continuous if the following holds:*

$$\begin{pmatrix} \forall \varepsilon > 0 & \exists \delta > 0 \\ \forall x \in \mathrm{dom}(f) & \exists y \in f(x) \end{pmatrix}\ \forall x' \in B(x,\delta) \cap \mathrm{dom}(f) \quad \exists y' \in B(y,\varepsilon) \cap f(x')\ . \tag{7}$$

f) *Some* $g :\subseteq X \rightrightarrows Y$ *tightens* f *(and* f *loosens* g*)*
if both $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ *and* $\forall x \in \mathrm{dom}(f) : g(x) \subseteq f(x)$ *hold.*

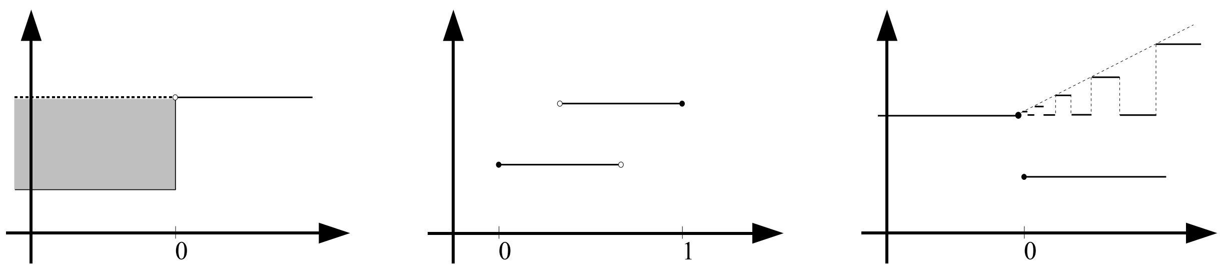


Fig. 2. a) Example of a uniformly weakly continuous but not weakly continuous relation. b) A semi-uniformly strongly continuous relation which is not uniformly strongly continuous. c) A compact, weakly and uniformly weakly continuous relation which is not computable relative to any oracle.

Lemma 3.2. *a) Let f be uniformly weakly continuous and suppose that f is pointwise compact in the sense that $f(x) \subseteq Y$ is compact for every $x \in X$. Then f is weakly continuous.*

b) Let f be nonuniformly weakly continuous and $\mathrm{dom}(f)$ *compact. Then f is uniformly weakly continuous.*

c) If f is Henkin-continuous and tightens g, then also g is Henkin-continuous.

d) If f and $g :\subseteq Y \rightrightarrows Z$ are Henkin-continuous, then so is $g \circ f :\subseteq X \rightrightarrows Z$.

e) A function $F :\subseteq \{\mathtt{0},\mathtt{1}\}^\omega \to \{\mathtt{0},\mathtt{1}\}^\omega$ is an (α,β)–realizer of f iff F tightens $\beta^{-1} \circ f \circ \alpha$ iff $\beta \circ F \circ \alpha^{-1}$ tightens f.

f) If $\mathrm{range}(f) \subseteq \mathrm{dom}(g)$ *holds and if both f and g map compact sets to compact sets, then so does $g \circ f$.*

Proposition 3.3. *a) The inverse $\rho_b^{-1} : [0,1] \rightrightarrows \{\mathtt{0},\mathtt{1}\}^\omega$ of the binary representation restricted to $[0,1]$ is not weakly continuous.*

b) Every $x \in \mathbb{R}$ has a signed digit expansion

$$x \;=\; \sum_{n=-N}^{\infty} a_n 2^{-n}, \qquad a_n \in \{\mathtt{0},\mathtt{1},\bar{\mathtt{1}}\} \tag{8}$$

with no *consecutive digit pair $\mathtt{11}$ nor $\bar{\mathtt{1}}\bar{\mathtt{1}}$ nor $\mathtt{1}\bar{\mathtt{1}}$ nor $\bar{\mathtt{1}}\mathtt{1}$.*

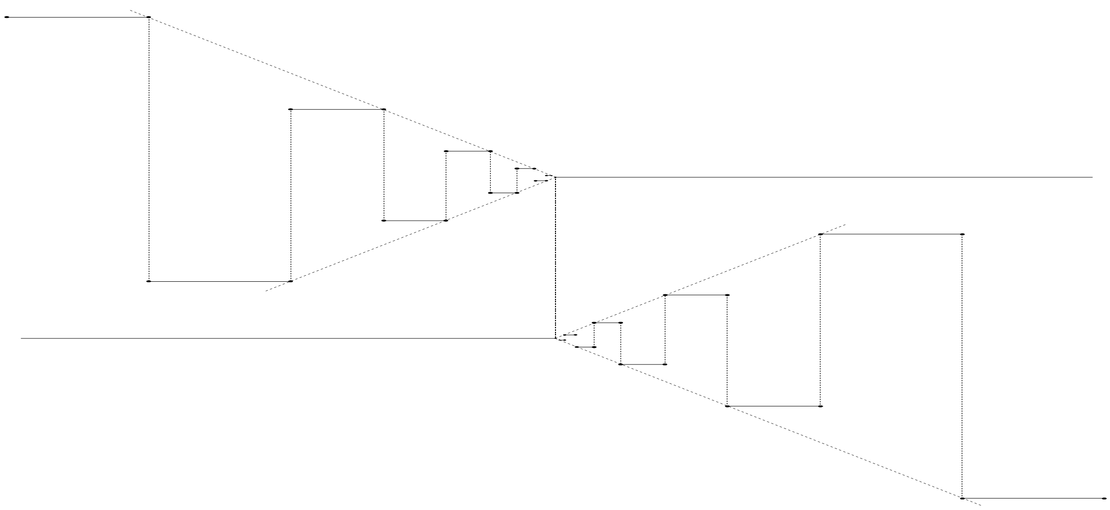
c) For $k \in \mathbb{N}$, each $|x| \leq \frac{2}{3} \cdot 2^{-k}$ admits such an expansion with $a_n = 0$ for all $n \leq k$. And, conversely, $x = \sum_{n=k+1}^{\infty} a_n 2^{-n}$ with $(a_n, a_{n+1}) \in \{\mathtt{10},\bar{\mathtt{1}}\mathtt{0},\mathtt{01},\mathtt{0}\bar{\mathtt{1}},\mathtt{00}\}$ for every n requires $|x| \leq \frac{2}{3} \cdot 2^{-k}$.

d) Let $x = \sum_{n=-N}^{\infty} a_n 2^{-n}$ be a signed digit expansion and $k \in \mathbb{N}$ such that $(a_n, a_{n+1}) \in \{\mathtt{10},\bar{\mathtt{1}}\mathtt{0},\mathtt{01},\mathtt{0}\bar{\mathtt{1}},\mathtt{00}\}$ for each $n > k$. Then every $x' \in [x - 2^{-k}/3, x + 2^{-k}/3]$ admits a signed digit expansion $x' = \sum_{n=-N}^{\infty} b_n 2^{-n}$ with $a_n = b_n \forall n \leq k$.

d) Let $\Sigma := \{\mathtt{0},\mathtt{1},\bar{\mathtt{1}},\mathtt{.}\}$. The inverse $\rho_{sd}^{-1} : \mathbb{R} \rightrightarrows \Sigma^\omega$ of the signed digit representation is Henkin-continuous.

Theorem 3.4. *Let $K \subseteq \mathbb{R}$ be compact and $f : K \rightrightarrows \mathbb{R}$ computable relative to some oracle. Then f is Henkin-continuous.*

Example 3.5 *A compact total Henkin–continuous but not relatively computable relation. (Dashed lines indicate alignment and are not part of the graph)*

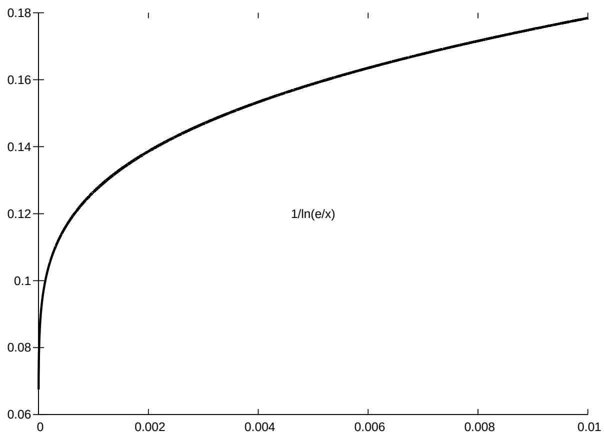


4 Computational Complexity

Definition 4.1. *Call $f : [0,1] \to \mathbb{R}$ computable in time $t(n)$ and space $s(n)$ if some Turing machine can, upon input of every ρ_{sd}–name of every $x \in \mathrm{dom}(f)$ and of n in unary, produce within these ressource bounds some $c \in \mathbb{Z}$ such that $|f(x) - c/2^{n+1}| \le 2^{-n}$.*

Lemma 4.2. *If f is (even oracle-) $(\rho_{\mathbb{D}}, \rho_{\mathbb{D}})$–computable in time $t(n)$, then $\mu : \mathbb{N} \ni n \mapsto t(n+2) \in \mathbb{N}$ constitutes a modulus of uniform continuity to f, i.e., $|x - x'| \le 2^{-\mu(n)} \Rightarrow |f(x) - f(x')| \le 2^{-n}$.*

Example 4.3 *The following function is computable in exponential time, but not in polynomial time — and oracles do not help:* $f : (0,1] \ni x \mapsto 1/\ln(e/x) \in (0,1], \quad f(0) = 0.$



	RECURSION THEORETIC	TOPOLOGICAL
NON-COMPUTABLE	$f(x) \equiv y[H]$	$f(x) = \mathrm{sgn}(x)$
EXPONENTIAL COMPLEXITY	$f(x) \equiv y[E]$	$f(x) = 1/\ln(e/x)$

Fig. 3. a) (Part of) the graph of $f(x) = 1/\ln(e/x)$ from Example 4.3 demonstrating its exponential rise from 0.
b) Lower bound techniques in real function computation; $H \subseteq \mathbb{N}$ is the Halting problem and $\mathbb{N} \supseteq E \in \mathsf{EXP} \setminus \mathcal{P}$.

In particular functional evaluation $(f, x) \mapsto f(x)$ is not computable within time bounded only in n, the output precision, even when restricting to smooth functions $f : [0,1] \to [0,1]$.

5 Recap on Blum-Shub-Smale (BSS) Machines

A BSS machine $\mathbb{M}$ (over $\mathbb{R}$) can in each step add, subtract, multiply, divide, and branch on the result of comparing two reals. Its memory consists of an infinite sequence of cells, each capable of holding a real number and accessed via two special index registers (similar to a two-head Turing machine). A program for $\mathbb{M}$ may store a finite number of real constants. The notions of *decidability* and *semi-decidability* translate straightforwardly from discrete $L \subseteq \{0,1\}^*$ and $L \subseteq \mathbb{N}^*$ to real languages $\mathbb{L} \subseteq \mathbb{R}^*$. Computing a function $f :\subseteq \mathbb{R}^* \to \mathbb{R}^*$ means that the machine, given $x \in \mathrm{dom}(f)$, outputs $f(x)$ within finitely many steps and terminates while diverging on inputs $x \notin \mathrm{dom}(f)$.

Example 5.1 *a)* $\mathrm{rank} : \mathbb{R}^{n\times m} \to \mathbb{N}$ *is uniformly BSS–computable (in time $\mathcal{O}(n^3+m^3)$)*

b) The multivalued mapping $\mathbb{R}^{n\times m} \ni A \rightrightarrows \{(b_1,\ldots) \text{ basis of } \mathrm{kern}(A)\} \in \mathbb{R}^{m\times *}$ is uniformly BSS–computable (in time $\mathcal{O}(n^3+m^3)$).

c) The multivalued mapping $\mathbb{R}^{n\times m} \ni A \rightrightarrows \{(c_1,\ldots) \text{ basis of } \mathrm{range}(A)\} \in \mathbb{R}^{n\times *}$ is uniformly BSS–computable (in time $\mathcal{O}(n^3+m^3)$).

d) The graph of the square root function is BSS–decidable.

e) $\mathbb{Q}$ is BSS semi-decidable; and so is the set $\mathbb{A}$ of algebraic reals.

f) The algebraic degree function $\deg : \mathbb{A} \to \mathbb{N}$ *is BSS–computable.*

g) A language $\mathbb{L} \subseteq \mathbb{R}^*$ is BSS semi-decidable iff $\mathbb{L} = \mathrm{range}(f)$ for some total computable $f : \mathbb{R}^* \to \mathbb{R}^*$.

h) The real Halting problem $\mathbb{H}$ is not BSS–decidable, where

$$\mathbb{H} := \{\langle \mathbb{M}, x\rangle : \text{BSS machine } \mathbb{M} \text{ terminates on input } x\}$$

Definition 5.2. *Fix a field $F \subseteq \mathbb{R}$ and $d \in \mathbb{N}$. A set*

$$\mathbb{B} \quad = \quad \{x \in \mathbb{R}^d \ : \ p_1(x) = \ldots = p_k(x) = 0 \ \wedge \ q_1(x) > 0 \wedge \ldots \wedge q_\ell(x) > 0\} \tag{9}$$

of solutions to a finite system of polynomial (in)equalities with $p_1,\ldots,p_k,q_1,\ldots,q_\ell \in F[X_1,\ldots,X_d]$ is called basic semi-algebraic over F.
A subset of $\mathbb{R}^d$ semi-algebraic over F *is a finite union of ones that are basic semi-algebraic over F. It is* countably semi-algebraic over F *if the union involves countably many members, all being basic semi-algebraic over F.*

If is known that every basic semi-algebraic set has at most finitely many connected components.

Lemma 5.3. *For $f :\subseteq \mathbb{R}^* \to \mathbb{R}^*$, and $c_1,\ldots,c_j \in \mathbb{R}$, consider the following claims:*

a) f is computable by a BSS Machine with constants $c_1,\ldots,c_j \in \mathbb{R}$.

b) There is an integer sequence $(d_n)_n$ such that $\mathrm{dom}(f) = \biguplus_n \mathbb{B}_n$ is the countable disjoint union of sets $\mathbb{B}_n \subseteq \mathbb{R}^{d_n}$ semi-algebraic over field extension $F := \mathbb{Q}(c_1,\ldots,c_j)$, and each restriction $f\big|_{\mathbb{B}_n}$, $n \in \mathbb{N}$, a quolynomial with coefficients from F.

c) There exists $c_{j+1} \in \mathbb{R}$ such that f is computable by a BSS Machine with constants $c_1,\ldots,c_j,c_{j+1}$.

Then a) implies b) implies c).

Corollary 5.4. *a) The square root function* $[0,\infty) \ni x \mapsto \sqrt{x} \geq 0$ *is not BSS–computable.*
b) The sequence $\mathbb{N} \ni n \mapsto e^{\sqrt{n}}$ *is not BSS–computable.*
c) $\mathbb{Q}$ *and* $\mathbb{A}$ *are not BSS–decidable*
d) nor is real integer linear programming $\{\langle A,b\rangle \mid A \in \mathbb{R}^{n\times m}, b \in \mathbb{Z}^m, \exists x \in \mathbb{Z}^n : A\cdot x = b\}$.

Fact 5.5 (Lindemann–Weierstraß) *Let* $a_1,\ldots,a_n$ *be algebraic yet linearly independent over* $\mathbb{Q}$. *Then* $e^{a_1},\ldots,e^{a_n}$ *are algebraically independent over* $\mathbb{Q}$.

6 Post's Problem over the Reals

Proposition 6.1. *a) Let* $x \in \mathbb{R}$, $\varepsilon > 0$, $N \in \mathbb{N}$. *There exists* $a \in \mathbb{A}$ *of* $\deg(a) = N$ *with* $|x-a| < \varepsilon$.
b) Let $f : \mathrm{dom}(f) \subseteq \mathbb{R} \to \mathbb{R}$ *be analytic and non-constant,* $T \subseteq \mathrm{dom}(f)$ *uncountable. Then,* f *maps some* $x \in T$ *to a transcendental value, that is,* $f(x) \notin \mathbb{A}$.
c) Fix non-constant $f = p/q \in \mathbb{R}(X)$ *with polynomials* p,q *of* $\deg(p) < n$, $\deg(p) < m$. *Let* $a_1,\ldots,a_{n+m} \in \mathrm{dom}(f)$ *be distinct real algebraic numbers with* $f(a_1),\ldots,f(a_{n+m}) \in \mathbb{Q}$. *There are co-prime polynomials* $\tilde{p},\tilde{q}$ *of* $\deg(\tilde{p}) < n$, $\deg(\tilde{q}) < m$ *with coefficients in the algebraic field extension* $\mathbb{Q}(a_1,\ldots,a_{n+m})$ *such that, for all* $x \in \mathrm{dom}(f) = \{x : q(x) \neq 0\} \subseteq \mathbb{R}$, *it holds* $f(x) = \tilde{f}(x) := \tilde{p}(x)/\tilde{q}(x)$.
d) Continuing c), let $d \geq \max_i \deg(a_i)$. *Then* $f(x) \notin \mathbb{Q}$ *for all transcendental* $x \in \mathrm{dom}(f)$ *as well as for all* $x \in \mathbb{A}$ *of* $\deg(x) > D := d^{n+m} \cdot \max\{n-1, m-1\}$.

Theorem 6.2. *The set* $\mathbb{Q}$ *of rationals is semi-decidable and undecidable yet strictly 'easier' than* $\mathbb{H}$*:* $\mathbb{A}$ *remains undecidable to a machine with oracle access to* $\mathbb{Q}$.

7 Computable Analysis vs. Algebraic Computability

Theorem 7.1. *a) Let* $f :\subseteq \mathbb{R}^k \to \mathbb{R}$ *be continuous and computable by a BSS machine* $\mathcal{M}$ *without real constants. Then* f *is* $(\rho^k \to \rho)$*–computable with oracle access to the Halting problem.*
b) To every ℓ *there exists a* C^ℓ *total function* $f : [0,1] \to \mathbb{R}$ *computable by a constant-free BSS machine which is not* $(\rho \to \rho)$*–computable.*

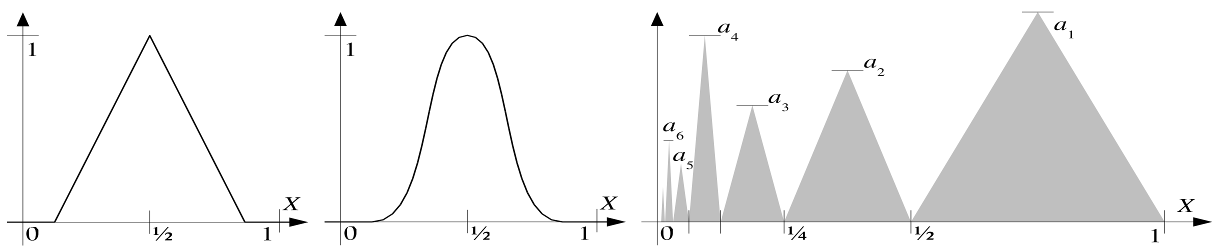


Fig. 4. A piecewise linear and a C^k unit pulse, and a non-overlapping superposition by scaled shifts

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