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Algebraic, Topological, and Physical Aspects of Computing

SS 2012, Exercise Sheet #11

Recall the definition of the degree $[E:F]=\dim_F(E)$ of a field extension E over F.
BESICOVITCH has proven that

$$\left[\mathbb{Q}\left(\sqrt[N_1]{p_1},\sqrt[N_2]{p_2},\ldots,\sqrt[N_d]{p_d}\right):\mathbb{Q}\right] \quad = \quad N_1\cdot N_2\cdots N_d \;;$$

cf. [Bes40, THEOREM 2] and see also [Alb03, bottom of p.2].
For fields $\mathbb{Q}\subseteq\mathbb{F}\subseteq\mathbb{R}$ and $0<r\in\mathbb{Q}$, let

$$\mathbb{F}(\sqrt[*]{r}) \quad := \quad \mathbb{F}\left(\left\{r^{\frac{1}{n}}:n\in\mathbb{N}\right\}\right)$$

where the corresponding fractional powers are understood as positive real numbers.
The goal of this exercise is to prove that $\mathbb{Q}(\sqrt[*]{2})$ and $\mathbb{Q}(\sqrt[*]{3})$ are
semi-decidable, undecidable, and incomparable.

EXERCISE 11:

a) If $(\frac{r}{s})^{1/n}\in\mathbb{Q}$ for $n\in\mathbb{N}$ and coprime $r,s\in\mathbb{N}$, then $r^{1/n},s^{1/n}\in\mathbb{N}$.

b) For $n_1,\ldots,n_k\in\mathbb{N}$ and squarefree $t\in\mathbb{N}$, $\mathbb{F}\left(\sqrt[n_1]{t},\ldots,\sqrt[n_k]{t}\right)=\mathbb{F}\left(\sqrt[N]{t}\right)$ where $N:=\mathrm{lcm}(n_1,\ldots,n_k)$ denotes the *least common multiple*.

c) For distinct prime numbers $p_1,\ldots,p_d,p_{d+1}$ and $n\in\mathbb{N}$, it holds

$$\left[\mathbb{Q}\left(\sqrt[*]{p_1},\sqrt[*]{p_2},\ldots,\sqrt[*]{p_d},\sqrt[*]{p_{d+1}}\right):\mathbb{Q}\left(\sqrt[*]{p_1},\sqrt[*]{p_2},\ldots,\sqrt[*]{p_d}\right)\right] \quad = \quad \infty \;.$$

d) To any $n\in\mathbb{N}$, $\varepsilon>0$, and $x\in\mathbb{R}$, there exists $y\in\mathbb{Q}(\sqrt[*]{2})$ of degree at least n over $\mathbb{Q}(\sqrt[*]{3})$ such that $|x-y|<\varepsilon$. Let $f\in\mathbb{R}(X)$, $f=\frac{p}{q}$ with polynomials p,q of degree less than n and m, respectively. Let $a_1,\ldots,a_{n+m}\in\mathbb{Q}(\sqrt[*]{2})\cap\mathrm{dom}(f)$ be distinct with $f(a_i)\in\mathbb{Q}(\sqrt[*]{3})$.

e) There are co-prime polynomials $\tilde{p},\tilde{q}$ of $\deg(\tilde{p})<n$, $\deg(\tilde{q})<m$ with coefficients in the algebraic field extension $\mathbb{Q}(\sqrt[*]{3};a_1,\ldots,a_{n+m})$ such that, for all $x\in\mathrm{dom}(f)=\{x:q(x)\neq 0\}\subseteq\mathbb{R}$, it holds $f(x)=\tilde{f}(x):=\tilde{p}(x)/\tilde{q}(x)$.

f) Let $d:=\max_i \deg_{\mathbb{Q}(\sqrt[*]{3})}(a_i)$. Then $f(x)\notin\mathbb{Q}(\sqrt[*]{3})$ for all transcendental $x\in\mathrm{dom}(f)$ as well as for all $x\in\mathbb{Q}(\sqrt[*]{2})$ of $\deg_{\mathbb{Q}(\sqrt[*]{3})}(x)>D:=d^{n+m}\cdot\max\{n-1,m-1\}$.

Literatur

[Alb03] T. ALBU: "*Cogalois Theory*", Dekker (2003).

[Bes40] A.S. BESICOVITCH: "On the Linear Independence of Fractional Powers of Integers", pp.3-6 in *J. London Math. Soc.* vol.**15** (1940).