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Algebraic, Topological, and Physical Aspects of Computing

SS 2012, Exercise Sheet #7

EXERCISE 7:
Recall that a function $f : X \to Y$ between topological spaces is called *proper* if the preimage $f^{-1}[K] \subseteq X$ is compact for every compact $K \subseteq Y$.
For a metric space (X,d), (X,d') induces the same topology where $d'(x,y) := \min\{d(x,y),1\}$. The *product topology* on $\hat{X} := \prod_n (X_n, d_n)$ is induced by $\hat{d} : \big((x_n)_n, (y_n)_n\big) := \sup_n d'_n(x_n,y_n)/2^n$.

a) Prove that the Cauchy representation $\rho_C :\subseteq (\mathbb{Q}^2)^\omega \to \mathbb{R}$ maps compact sets to compact sets, where $\rho_C : (q_n,\varepsilon)_n \mapsto x$ whenever $|x-q_n| \leq \varepsilon_n \to 0$ as $n \to \infty$.
Here $(\mathbb{Q}^2)^\omega$ is considered equipped with the product topology.

b) Repeat for the dyadic representation $\rho :\subseteq \mathbb{Z}^\omega \to \mathbb{R}$,
i.e. $\rho : (c_n)_n \mapsto x$ whenever $|x - c_n/2^{n+1}| \leq 2^{-n}$.

c) Is ρ_C proper? Prove or disprove!

d) Is ρ proper? Prove or disprove!

e) Prove: $\quad \forall x \in \mathbb{R} \quad \exists \bar{c} = (c_n)_n \in \mathbb{Z}^\omega \quad \forall m \in \mathbb{N} \quad \forall |x-x'| \leq 2^{-m-1} \quad \exists \vec{c}' \in \mathbb{Z}^\omega :$
$$\rho(\bar{c}) = x \ \wedge\ \rho(\vec{c}') = x' \ \wedge\ c_1 = c'_1 \ \wedge \ldots \wedge\ c_m = c'_m.$$