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Algebraic, Topological, and Physical Aspects of Computing

SS 2012, Exercise Sheet #5

EXERCISE 5:

a) Prove that, for each $1 \leq j \leq N$, the following set $\mathrm{Ext}_{N,d,j}$ is open and even $\rho^{N\times d}$–r.e:

$$\big\{(\vec{x}_1,\ldots,\vec{x}_N) : \vec{x}_1,\ldots,\vec{x}_N \in \mathbb{R}^d, \vec{x}_j \text{ extreme point of } \operatorname{chull}(\vec{x}_1,\ldots,\vec{x}_N)\big\}$$

Hint: For a polytope P, $\vec{x} \in P$ is extreme iff it is *exposed* in the sense that there exists an affine hyperplane H with $\{\vec{x}\} = P \cap H$.

b) Prove that closed function image, i.e. the following mapping, is $([\rho^d \to \rho^k] \times \psi^d_<, \psi^k_<)$–computable:

$$C(\mathbb{R}^d \to \mathbb{R}^k) \times \mathcal{A}^{(d)} \ni (f, A) \mapsto \overline{f[A]} \in \mathcal{A}^{(k)}$$

c) Prove that $f : [0,1]^d \to \mathbb{R}$ is computable iff

i) it admits a recursive global modulus of continuity*
ii) and the sequence $\big(f(\vec{q})\big)_{\vec{q}\in\mathbb{D}^d\cap[0,1]^d}$ is computable.

d) Prove that the mapping $C(\mathbb{R}^d) \ni f \mapsto f^{-1}[0] \in \mathcal{A}^{(d)}$ is $([\rho^d \to \rho], \psi^d_>)$–computable.
In fact $C(\mathbb{R}^d \to \mathbb{R}^k) \times \mathcal{A}^{(k)} \ni (f, B) \mapsto f^{-1}[B] \in \mathcal{A}^{(d)}$ is $([\rho^d \to \rho^k] \times \psi^k_>, \psi^d_>)$–computable.

*i.e., a mapping $\mu : \mathbb{N} \to \mathbb{N}$ with $|x-y| \leq 2^{-\mu(n)} \;\Rightarrow\; |f(x) - f(y)| \leq 2^{-n}$