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Algebraic, Topological, and Physical Aspects of Computing

SS 2012, Exercise Sheet #2

EXERCISE 1:

f) Show that every $(\rho_n \to \rho_n)$–computable $f : \mathbb{R} \to \mathbb{R}$ is continuous.

g) Formalize real computation by approximation up to relative error which for every fixed number coincides with (e.g. Cauchy) computability; show that addition is not computable in this sense.

EXERCISE 2:

Call $\vec{u} \in \{0,1\}^*$ an initial segment of $\vec{v} \in \{0,1\}^*$ (and write "$\vec{u} \sqsubseteq \vec{v}$") if there exists $\vec{w} \in \{0,1\}^*$ with $\vec{v} = \vec{u} \circ \vec{w}$. Similarly write "$\vec{u} \sqsubseteq \bar{v}$" if $\bar{v} = \vec{u} \circ \bar{w}$ for $\bar{v} \in \{0,1\}^\omega$ and some $\bar{w} \in \{0,1\}^\omega$. For $\bar{\sigma} \in \{0,1\}^\omega$ abbreviate $\bar{\sigma}|_{\leq n} := (\sigma_1, \ldots, \sigma_n) \in \{0,1\}^n$. We say that $f : \{0,1\}^* \to \{0,1\}^*$ is monotone if it holds

$$\forall \vec{u}, \vec{v} : \quad \vec{u} \sqsubseteq \vec{v} \;\Rightarrow\; f(\vec{u}) \sqsubseteq f(\vec{v}) \;.$$

a) Suppose f is also unbounded on $\bar{\sigma} \in \{0,1\}^\omega$ in that $\lim_n |f(\bar{\sigma}|_{\leq n})| = \infty$. Show that there exists precisely one $\bar{\tau} \in \{0,1\}^\omega$ (denoted by $\tau = \sup_n f(\bar{\sigma}|_{\leq n})$) with $\forall n : f(\bar{\sigma}_{\leq n}) \sqsubseteq \bar{\tau}$.

b) Suppose f is monotone. Prove that the following function $f_\omega :\subseteq \{0,1\}^\omega \to \{0,1\}^\omega$ is continuous:

$$\mathrm{dom}(f_\omega) = \{\bar{\sigma} \mid f \text{ unbounded on } \bar{\sigma}\}, \qquad f_\omega : \bar{\sigma} \mapsto \sup_n f(\bar{\sigma}|_{\leq n})$$

c) Suppose $F :\subseteq \{0,1\}^\omega \to \{0,1\}^\omega$ is continuous. Construct some monotone f with $f_\omega\big|_{\mathrm{dom}(F)} = F$.

d) Every $(\rho \to \rho_<)$–computable $f : [0,1] \to \mathbb{R}$ is lower semi-continuous.