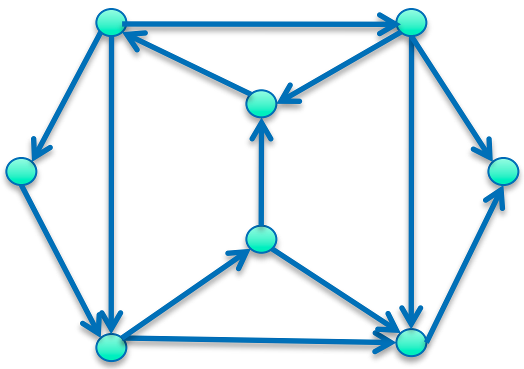
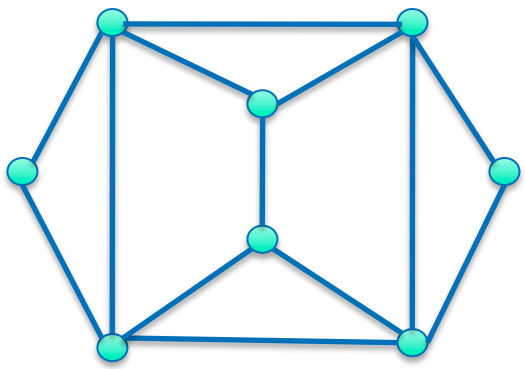
Graphs and classic graph problems

- What is a graph?
  - An undirected **graph** is a pair G=(V,E), with E ⊆{{u,v} | u,v∈ V}.
    - The elememts of E are not ordered.
    - Elements of V are called **nodes, or vertices resp.**
      Elements of E are called **edges**
  - A **directed graph** is a pair G=(V,E) as well. However, the elements of E are ordered pairs of elements of V. Thus
    E ⊆{(u,v) | u,v∈ V}.
    - Elements of V are called nodes
      Elements of E are called **directed edges** (in ger.: **gerichtete Kanten** or **Bögen**)



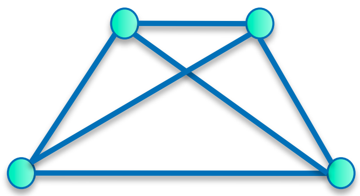


Neighborhood relations

Incidence: A node v is said to be incident to an edge e if : v∈e.

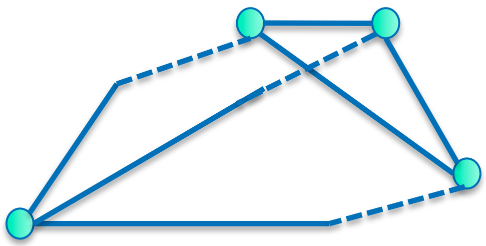
adjacency: Two nodes are called adjacent in G, if {x,y}∈E.

Degree: The degree of a node v [deg(v)] is the number of with v incident edges.
Clique: If any two nodes of G a neighbors, G is called the complete graph.
G is also called a clique.



Graphs and classic graph problems

Claim: $\sum_{v \in V} \deg(v) = 2|E|$



Proof: counting edges

Claim: {v | deg(v) odd} is even

Proof:clear

Graphs and classic graph problems

TECHNISCHE UNIVERSITÄT DARMSTADT

- Paths and Cycles

  – A **path** in a graph (directed or non-directed) $G = (V,E)$ is a sequence of nodes $(v_0,v_1,...,v_k)$ from V, such that for all $i \in \{1,...,k\}$ is valid that $(v_{i-1},v_i) \in E$ (or $\{v_{i-1},v_i\} \in E$, resp.) is an edge.
  If $i<j$, v_i is called a predecessor of v_j, and v_j a successor of v_i

  – A path is called **simple**, if no node occurs more than once in the sequence $(v_1,...,v_k)$.

  – A simple path is called **cycle**, if $v_0 = v_k$.

  – A graph is called **acyclic** (ger.: kreisfrei), if there is no cycle contained in the graph.

Graphs and classic graph problems

TECHNISCHE UNIVERSITÄT DARMSTADT

- Connected components

  – A undirected graph G is **connected**, if there is a path from v to w for all node pairs v,w∈V.
  – A directed graph is strongly connected, if there are paths from all nodes to any other node.

  – Connected parts of G are called **connected components.**

Reminder: A set V with binary relation is called an equivalence relation if for all for all x,y,z ∈V:

i) x ~x, reflexive

ii) x~y => y~ x , symmetric

iii) x~y and y~z => x~z, transitive

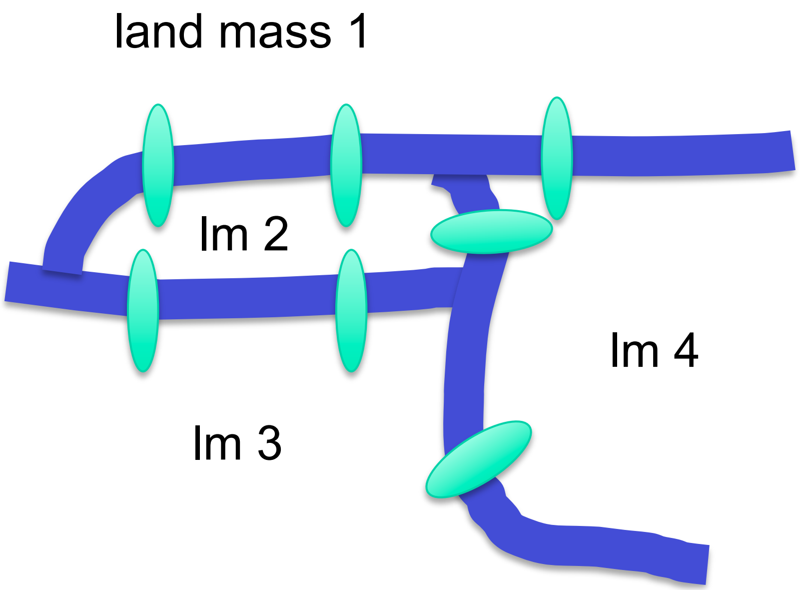
Observations: Connected components of a graph form equivalence classes.

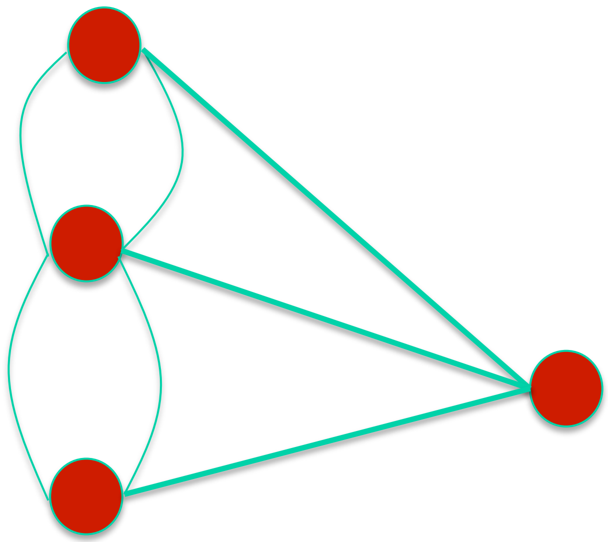
G is connected if it has only one equivalence class.

Graphs and classic graph problems

Seven bridges of Königsberg (Euler 1736):

Is it possible was to find a walk through the city that would cross each bridge once and only once? Such a walk is called *Eulerian path*.



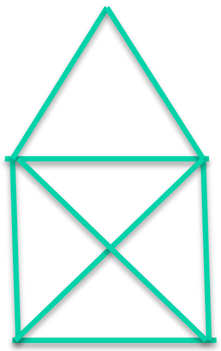


An Eulerian path with identical start and end point is called *Eulerian cycle*.
→ no, both do not exist in Königsberg of 1736.

Graphs and classic graph problems

Seven bridges of Königsberg (Euler 1736):

Claim: A necessary and sufficient condition for a walk of the desired form is that the graph is connected and has exactly zero (-> Eulerian cycle) or two nodes (-> Eulerian path) of odd degree.



Haus vom Nikolaus contains

- 1 node of degree 2
- 2 nodes of degree 4
- 2 nodes of degree 3

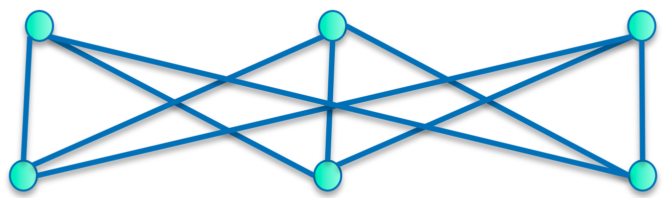
→ Contains an Eulerian path, but no Eulerian cycle.

Graphs and classic graph problems

TECHNISCHE UNIVERSITÄT DARMSTADT

Bipartite graphs

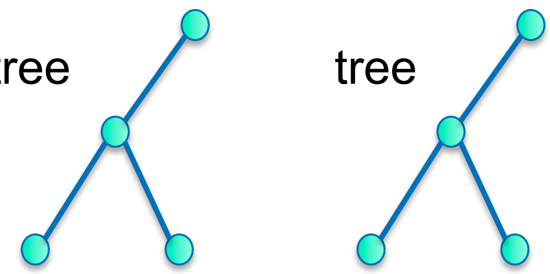
A graph is called bipartit, if it is possible to partition V into two disjoint parts V_1 and V_2 , such that each edge has one end in V_1 and the other one in V_2.



Claim: A graph is bipartite if and only if any cycle in the graph has even length.
Proof: Exercise

Graphs and classic graph problems

- A **tree** is a connected graph without any cycles
- A **forest** is a graph without any cyles, thus a collection of trees.
- Nodes with degree 1 are called **leaf**, the others **inner nodes**.



together: a forest

Claim: A forest F has $|E| = |V| – C(F)$ edges, with C number of components.

Claim: If $|E| > |V| – C(F)$, G will contain a cycle.

Claim: For any tree is valid: $|E| = |V| – 1$

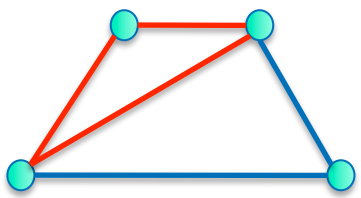
Proof: Exercise

Graphs and classic graph problems

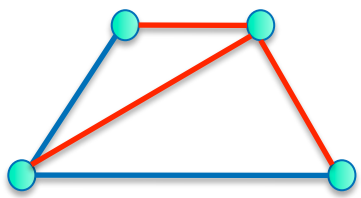
- subgraphs, spanning subgraphs

  – G'=(V',E') is called **subgraph** of G=(V,E), if V'⊆V and E'⊆E.

  – G' ⊆ G is called a **spanning subgraph** of G, when additionally: V'=V



Graph with subgraph



spanning tree

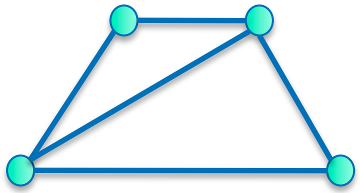
Claim: A graph is connected iff it contains a spanning tree.
(Proof: <=: clear; =>: remove edges from cycles)

Graphs and classic graph problems

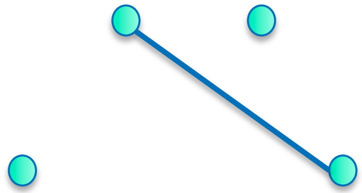
TECHNISCHE UNIVERSITÄT DARMSTADT

- **Complementary Graph**

  – The for G **complementary graph** G' is the graph
  G'=(V,E'), with $(i,j) \in E' \Leftrightarrow (i,j) \notin E$



graph



complement

  – Claim: At least one of the graphs G or G' is connected.
  Proof: Exercise.