Algorithmic Discrete Mathematics 1. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

Department of Mathematics
PD Dr. Ulf Lorenz
Dipl.-Math. David Meffert

SS 2012
18. and 19. April 2012
Version of April 26, 2012

Groupwork

Exercise G1 (Recurrence)
Given the following recurrence

$$T(1) = 0 \qquad T(2^k) = 2T(2^{k-1}) + 2^{k+1} - 1 \qquad \text{for } k > 0.$$

Prove that the formula $T(2^k) = 2 \cdot k \cdot 2^k - (2^k - 1)$ holds for every $k \geq 0$.

Solution: We use proof by induction over $k \in \mathbb{N}$. For $k = 0$ we have

$$T(1) = 2 \cdot 0 \cdot 1 - (1 - 1) = 0.$$

Assuming we have $T(2^k) = 2 \cdot k \cdot 2^k - (2^k - 1)$ for a $k \in \mathbb{N}$, we get

$$\begin{aligned}
T(2^{k+1}) &= 2 \cdot T(2^k) + 2^{k+2} - 1 \\
&= 2 \cdot (2 \cdot k \cdot 2^k - 2^k + 1) + 2^{k+2} - 1 \\
&= 2 \cdot k \cdot 2^{k+1} - 2^{k+1} + 1 + 2^{k+2} \\
&= 2 \cdot (k \cdot 2^{k+1} + 2^{k+1}) - 2^{k+1} + 1 \\
&= 2(k+1)2^{k+1} - (2^{k+1} - 1).
\end{aligned}$$

Exercise G2 (Binomial coefficients)
Binomial coefficients play an important role in combinatorics. They describe the number of possibilities to choose k objects from a given set containing n objects (without putting objects back and without respecting the order of the objects). For $n \geq k$ the binomial coefficient is given by the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

- Show that

  $$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

  holds for $n > k$.
- Now prove the formula

  $$\sum_{i=1}^{n} i = \binom{n}{2} + \binom{n}{1} = \frac{1}{2}n^2 + \frac{1}{2}n$$

  for $n \geq 2$.

Solution:

- First part: By definition we get

$$\binom{n}{k+1}+\binom{n}{k}=\frac{n!}{(k+1)!(n-k-1)!}+\frac{n!}{k!(n-k)!}=\frac{n!(n-k)+n!(k+1)}{(k+1)!(n-k)!}$$
$$=\frac{n!(n+1)}{(k+1)!(n-k)!}=\binom{n+1}{k+1}$$

- Second Part: We use induction over $n \geq 2$ to prove the formula. For $n=2$ we get

$$\sum_{i=1}^{2} i = 1+2 = 3$$

and

$$\binom{2}{2}+\binom{2}{1}=1+2=3=\frac{1}{2}2^2+\frac{1}{2}2.$$

Assuming we have

$$\sum_{i=1}^{n} i=\binom{n}{2}+\binom{n}{1}=\frac{1}{2}n^2+\frac{1}{2}n$$

for a $n \geq 2$ we get the first equality

$$\sum_{i=1}^{n+1} i=\sum_{i=1}^{n} i+(n+1)=\binom{n}{2}+\binom{n}{1}+\binom{n+1}{1}$$
$$=\binom{n+1}{2}+\binom{n+1}{1}$$

by additionally using the first part of this exercise in the last step. The second equality is proven by

$$\sum_{i=1}^{n+1} i=\frac{1}{2}n^2+\frac{1}{2}n+(n+1)=\frac{1}{2}n^2+n+\frac{1}{2}+\frac{1}{2}n+\frac{1}{2}$$
$$=\frac{1}{2}(n+1)^2+\frac{1}{2}(n+1).$$

Exercise G3 (Combinatorics)

(a) Max wants to take a picture of his 11 friends. Therefore he wants to align them in two different rows. How many possibilities has Max to do so, if he does not want any of the two rows to be empty?

(b) A bit may have to different states (0 and 1). A byte consists of 8 bits (e.g. 01101011). How many different bytes do exist?

(c) In a starcraft II tournament with 32 players participating, how many possibilities are there for
- the participants of the semifinals (= round of last 4)?
- the order of the first 4 places?

(d) How many different 'words' do you get by permuting the letters of the word MATHEMATICS?

Solution:

(a) Max has 10 possibilities to make two rows with 11 persons. Notice that although you have only 5 possible ways to sum up two natural numbers to 11, it makes a difference if there are for example 5 persons in the front row or on the back row. Furthermore there are 11! possibilities of ordering 11 persons, so we get $10 \cdot 11! = 399168000$ possibilities in total.

(b) There are $2^8 = 256$ different bytes.

(c)
- There are $\binom{32}{4} = 35960$ possible combinations of players.
- There are $32 \cdot 31 \cdot 30 \cdot 29 = 863040$ possible orders for the first 4 places.

(d) If a word with 11 letters contains only different letters there would be 11! different words by permuting the letters. So because the letters T, M and A appear twice we have to divide by $2 \cdot 2 \cdot 2 = 8$ and get $11!/8 = 4989600$ different words.

Exercise G4 (Sets)

(a) Given the sets $A = \{\text{red, green, blue}\}$ and $B = \{\text{blue, red, yellow}\}$. What is their union, intersection and symmetric difference?

(b) Name all subsets of A and enumerate them systematically. How many subsets do you get?

(c) Given three sets of the cardinalities 3, 6 and 9. How many elements do their union/intersection have at least/most.

(d) Given the three sets L, M, K. Prove the equation

$$(M \cap N) \cup L = (M \cup L) \cap (N \cup L),$$

by first drawing a picture and then proving it formally.

Remark: The *symmetric difference* $\triangle$ of two sets A and B is defined by

$$A \triangle B := (A \setminus B) \cup (B \setminus A).$$

Solution:

(a) We have $A \cup B = \{\text{red, green, blue, yellow}\}$, $A \cap B = \{\text{red, blue}\}$ and $A \triangle B = \{\text{green, yellow}\}$.

(b)

$$\begin{aligned}
0 &\Leftrightarrow 0_2 \Leftrightarrow 000 \Leftrightarrow \emptyset && (1)\\
1 &\Leftrightarrow 1_2 \Leftrightarrow 001 \Leftrightarrow \{\text{blue}\} && (2)\\
2 &\Leftrightarrow 10_2 \Leftrightarrow 010 \Leftrightarrow \{\text{green}\} && (3)\\
3 &\Leftrightarrow 11_2 \Leftrightarrow 011 \Leftrightarrow \{\text{green, blue}\} && (4)\\
4 &\Leftrightarrow 100_2 \Leftrightarrow 100 \Leftrightarrow \{\text{red}\} && (5)\\
5 &\Leftrightarrow 101_2 \Leftrightarrow 101 \Leftrightarrow \{\text{red, blue}\} && (6)\\
6 &\Leftrightarrow 110_2 \Leftrightarrow 110 \Leftrightarrow \{\text{red, green}\} && (7)\\
7 &\Leftrightarrow 111_2 \Leftrightarrow 111 \Leftrightarrow \{\text{red, green, blue}\} && (8)
\end{aligned}$$

We get $8 = 2^3$ subsets.

(c) The union contains at most 18 and at least 9 elements. These cases occur if the three sets are disjoint or included in each other respectively. The intersection contains at most 3 elements and can be empty. Of course it is empty if the three sets are disjoint and contains 3 elements if the sets are included in each other.

(d) We have

$$\begin{aligned}
x \in (M \cap N) \cup L &\iff (x \in M \text{ and } x \in N) \text{ or } x \in L\\
&\iff (x \in M \text{ or } x \in L) \text{ and } (x \in N \text{ or } x \in L)\\
&\iff x \in (M \cup L) \cap (N \cup L).
\end{aligned}$$

Homework

Exercise H1 (Combinatorics) (10 points)

(a) In the cafeteria there are 10 people waiting in one line.
- In how many different ways can they be lined up?
- Suppose 4 of them want to eat fish for lunch. How many different possibilities do you have to choose those 4 people?
- Suppose now that the fish eaters are directly lined up after each other. In how many different ways can the 10 people be lined up now?

(b) Starting in Wiesbaden we want to visit 6 of the 16 capitals of the German states. How many possible trips do we have?

(c) The frog Leo wants to advance on a strip of paper which is numbered by $|\,1\,|\,2\,|\,3\,|\,\ldots\,|\,n\,|$. He can can do that by either jumping two spaces or just one space. How many different ways of getting to the field with the number n does he have, if he starts on the field with the number 1.

Solution:

(a)
- There are $10! = 3628800$ possibilities to line them up.
- There are $\binom{10}{4}$ possibilities to choose those 4 people.
- There a 7 possible places where the 4 people can stand directly behind each other and because you can change their order we have to multiply this by $4!$. The missing 6 people can be lined up in $6!$ possible ways, so we get $7 \cdot 4! \cdot 6! = 120960$.

(b) Because we want start in Wiesbaden, one of the 6 cities in the trip is already determined. For the other 5 cities we have $15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 = 360360$ choices.

(c) We observe that on a strip of paper which has n fields Leo may do at most $\lfloor \frac{n-1}{2} \rfloor$ jumps of length 2. For every number k of those big jumps we have $\binom{n-(k+1)}{k}$ possible places on the strip Leo can do those big jumps. By summing up all those possibilities we get

$$\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-(k+1)}{k}.$$

Another approach involves the Fibonacci numbers. Therefore we define the n-th Fibonacci number by F_n (notice we set $F_1 = F_2 = 1$). For $n \geq 1$ we define J_n as the number of possible ways Leo can get to the field with the number n. We observe that one has two possbilities to do that. Either Leo can jump to field $n-2$ and do a big jump afterwards or he can jump to field $n-1$ doing a small jump afterwards. So we have the recursion

$$J_n = J_{n-1} + J_{n-2}$$

for $n \geq 3$. This is just the same recursion as for the Fibonacci numbers and we also have $F_1 = F_2 = J_1 = J_2 = 1$. This means we have $J_n = F_n$ for all $n \in \mathbb{N}$.

Exercise H2 (Symmetric differences) (10 points)

Let A, B be arbitrary sets.

(a) What is the symmetric difference of A and A?

(b) Prove the following equality of sets:

$$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$$

Remark: Do this by first showing $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$ and the other inclusion afterwards.

(c) Define $C := A \triangle B$ as the symmetric difference of the sets A and B. Now determine the symmetric difference of the sets A and C. Which set do you get? Write it down in a formula and prove it. It helps to draw a picture first.

Solution:

(a) The symmetric difference of two sets M_1, M_2 contains all elements which are contained in exactly one of sets M_1 and M_2, so we get $A \triangle A = \emptyset$.

(b) As remarked we show $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$ first. So let $x \in (A \cup B) \setminus (A \cap B)$. If we have $x \in A$, we can conclude $x \notin B$ because of $x \notin A \cap B$. So we have $x \in A \setminus B$. The same way we get $x \in B \setminus A$ for $x \in B$.

To show the other inclusion let $x \in (A \setminus B) \cup (B \setminus A)$. For $x \in A \setminus B$ we have $x \in A \subseteq A \cup B$. Additionally we have $x \notin A \cap B$ because of $x \notin B$. So we conclude $x \in (A \cup B) \setminus (A \cap B)$. For $x \in B \setminus A$ we can conclude this the same way.

(c) We claim

$$A \triangle C = A \triangle (A \triangle B) = B.$$

To prove this we write

$$A \triangle C = (A \cup (A \triangle B)) \setminus (A \cap (A \triangle B))$$

and show that this set is equal to B. So as in part (b) we show both inclusions. The given proof is rather short, so be sure you understand every step in detail. Let $x \in (A \cup (A \triangle B)) \setminus (A \cap (A \triangle B))$. By definition it is clear that we have $A \triangle C = (A \cup (A \triangle B)) \setminus (A \cap (A \triangle B)) \subseteq A \cup B$, so by showing $x \in A$ implies $x \in B$ we are already done with the first inclusion. So let $x \in A$. We will show $x \in B$ by contradiction, assuming $x \notin B$. By definition of $A \triangle B$ we can conclude $x \in A \triangle B$ then and therefore $x \in (A \cap (A \triangle B))$. This contradicts the original choice of $x \in (A \cup (A \triangle B)) \setminus (A \cap (A \triangle B))$, so we have $x \in B$. Now take a look at the other inclusion. Let $x \in B$. We distinguish the cases $x \in B \setminus A$ and $x \in B \cap A$.

For $x \in B \setminus A$ we get $x \in A \triangle B$ and $x \notin A$, so we conclude $x \in (A \cup (A \triangle B)) \setminus (A \cap (A \triangle B))$. For $x \in A \cap B$ we get $x \in A$ and $x \notin A \triangle B$ and therefore $x \in (A \cup (A \triangle B)) \setminus (A \cap (A \triangle B))$. This proves the second inclusion.

The solution to this excersis would have been much easier if we already knew that the symmetric difference is associative. Then we could just calculate

$$A \triangle (A \triangle B) = (A \triangle A) \triangle B = \emptyset \triangle B = B.$$

Exercise H3 (Binomial coefficients) (10 points)

Let $k, n \in \mathbb{N}$ with $k \leq n$.

(a) Prove the formula

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

i. by explaining it combinatorial,

ii. by algebraic calculation with help of the definition of the binomial coefficient.

(b) Prove the formula

$$\binom{n+1}{k+1} + \binom{n+1}{k} = \binom{n+2}{k+1}.$$

i. by explaining it combinatorial,

ii. by algebraic calculation with help of the definition of the binomial coefficient.

Solution:

(a) i. As we know the left side of the equation can be interpreted as the number of ways to choose k elements out of a set containing n elements. So we have the number of all subsets of cardinality k. Another way to chooses k elements (compare to right side of equation) is choosing one element (n possible choices) and then choosing $k-1$ elements of the remaing $n-1$ elements. By doing this we get each subset of cardinality k not just once but k-times. So we have to divide this number by k.

ii. By definition of the binomial coefficients we get

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)!}{k(k-1)!(n-1-(k-1))!} = \frac{n}{k}\binom{n-1}{k-1}.$$

(b) i. The right side of the equation coincides with the number of subsets of cardinality $k+1$ of a given set M with cardinality $n+2$. We can also count them by fixing one element $e \in M$ and counting the subsets which do not contain this element e (so choose $k+1$ elements of the remaining $n+1$ elements) and counting the sets which contain the fixed element e (choose k elements of the remaining $n+1$ elements). This coincides with the left side of the equation.

ii. By definition of the binomial coefficients we get

$$\begin{aligned}\binom{n+1}{k+1} + \binom{n+1}{k} &= \frac{(n+1)!}{(k+1)!(n-k)!} + \frac{(n+1)!}{k!(n-k+1)!} \\ &= \frac{(n+1)!(n-k+1) + (k+1)(n+1)!}{(k+1)!(n-k+1)!} \\ &= \frac{(n+2)!}{(k+1)!(n-k+1)!} \\ &= \binom{n+2}{k+1}.\end{aligned}$$