Algorithmic Discrete Mathematics
1. Exercise Sheet

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Groupwork

Exercise G1 (Recurrence)
Given the following recurrence

$$T(1) = 0 \qquad T(2^k) = 2T(2^{k-1}) + 2^{k+1} - 1 \qquad \text{for } k > 0.$$

Proof that the formula $T(2^k) = 2 \cdot k \cdot 2^k - (2^k - 1)$ holds for every $k \geq 0$.

Exercise G2 (Binomial coefficients)
Binomial coefficients play an important role in combinatorics. They describe the number of possibilities to choose k objects from a given set containing n objects (without putting objects back and without respecting the order of the objects). For $n \geq k$ the binomial coefficient is given by the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

- Show that

  $$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

  holds for $n > k$.
- Now proof the formula

  $$\sum_{i=1}^{n} i = \binom{n}{2} + \binom{n}{1} = \frac{1}{2}n^2 + \frac{1}{2}n$$

  for $n \geq 2$.

Exercise G3 (Combinatorics)

(a) Max wants to take a picture of his 11 friends. Therefore he wants to align them in two different rows. How many possibilities has Max to do so, if he does not want any of the two rows to be empty?

(b) A bit may have to different states (0 and 1). A byte consists of 8 bits (e.g. 01101011). How many different bytes do exist?

(c) In a starcraft II tournament with 32 players participating, how many possibilities are there for
- the participants of the semifinals (= round of last 4)?
- the order of the first 4 places?

(d) How many different 'words' do you get by permuting the letters of the word MATHEMATICS?

Exercise G4 (Sets)

(a) Given the sets $A = \{\text{red, green, blue}\}$ and $B = \{\text{blue, red, yellow}\}$. What is their union, intersection and symmetric difference?

(b) Name all subsets of A and enumerate them systematically. How many subsets do you get?

(c) Given three sets of the cardinalities 3, 6 and 9. How many elements do their union/intersection have at least/most.

(d) Given the three sets L, M, K. Proof the equation

$$(M \cap N) \cup L = (M \cup L) \cap (N \cup L),$$

by first drawing a picture and then proofing it formally.

Remark: The *symmetric difference* $\triangle$ of two sets A and B is defined by

$$A \triangle B := (A \setminus B) \cup (B \setminus A).$$

Homework

Exercise H1 (Combinatorics) (10 points)

(a) In the cafeteria there are 10 people waiting in one line.
- In how many different ways can they be lined up?
- Suppose 4 of them want to eat fish for lunch. How many different possibilities do you have to choose those 4 people?
- Suppose now that the fish eaters are directly lined up after each other. In how many different ways can the 10 people be lined up now?

(b) Starting in Wiesbaden we want to visit 6 of the 16 capitals of the German states. How many possible trips do we have?

(c) The frog Leo wants to advance on a strip of paper which is numbered by $| \, 1 \, | \, 2 \, | \, 3 \, | \dots | \, n \, |$. He can can do that by either jumping two spaces or just one space. How many different ways of getting to the field with the number n does he have, if he starts on the field with the number 1.

Exercise H2 (Symmetric differences) (10 points)

Let A, B be arbitrary sets.

(a) What is the symmetric difference of A and A?

(b) Proof the following equality of sets:

$$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$$

Remark: Do this by first showing $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$ and the other inclusion afterwards.

(c) Define $C := A \triangle B$ as the symmetric difference of the sets A and B. Now determine the symmetric difference of the sets A and C. Which set do you get? Write it down in a formula and proof it. It helps to draw a picture first.

Exercise H3 (Binomial coefficients) (10 points)

Let $k, n \in \mathbb{N}$ with $k \leq n$.

(a) Proof the formula

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

i. by explaining it combinatorial,
ii. by algebraic calculation with help of the definition of the binomial coefficient.

(b) Proof the formula

$$\binom{n+1}{k+1} + \binom{n+1}{k} = \binom{n+2}{k+1}.$$

i. by explaining it combinatorial,
ii. by algebraic calculation with help of the definition of the binomial coefficient.