Analysis III – Complex Analysis
Hints for solution for the
8. Exercise Sheet

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Groupwork

Exercise G1 (A strange Laurent series expansion)

Consider the following Laurent series expansion of the zero function:

$$\begin{aligned} 0 &= \frac{1}{z-1} + \frac{1}{1-z} = \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} + \frac{1}{1-z} \\ &= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} z^n = \sum_{n=-\infty}^{\infty} z^n. \end{aligned}$$

This contradicts the uniqueness of the Laurent series expansion, doesn't it?

Hints for solution: The identity above only holds on $\{z \in \mathbb{C} : |z| < 1\} \cap \{z \in \mathbb{C} : |z| > 1\} = \emptyset$. Thus the epxansion above is meaningless and doesn't contradict the uniqueness of the Laurent series expansion.

Exercise G2 (Some Laurent series expansions)

Consider the holomorphic function $f: \mathbb{C}\setminus\{1,3\} \to \mathbb{C}$, $f(z) = \frac{2}{z^2-4z+3}$. Use the partial fraction decomposition

$$f(z) = \frac{1}{1-z} + \frac{1}{z-3}$$

to expand f on the following annuli into a Laurent series in $z_0 = 0$:

$$R_1 := \{z \in \mathbb{C} : \; 0 < |z| < 1\}, \quad R_2 := \{z \in \mathbb{C} : \; 1 < |z| < 3\}, \quad R_3 := \{z \in \mathbb{C} : \; 3 < |z| < 42\}.$$

Hints for solution: We use the expansion into the geometric series. On R_1 we get

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \qquad \text{and} \qquad \frac{1}{z-3} = -\frac{1}{3}\cdot\frac{1}{1-\frac{z}{3}} = -\sum_{k=0}^{\infty}\frac{1}{3^{k+1}}z^k.$$

This leads to

$$f(z) = \sum_{k=0}^{\infty}\left(1-\frac{1}{3^{k+1}}\right)\cdot z^k.$$

Of course we get the power series expansion of f which converges on $K_1(0)$.
For $|z| > 1$ we use

$$\frac{1}{1-z} = -\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} = -\sum_{k=1}^{\infty} z^{-k}.$$

This leads on R_2 to the Laurent series expansion

$$f(z) = -\sum_{k=1}^{\infty} z^{-k} - \sum_{k=0}^{\infty}\frac{1}{3^{k+1}}z^k.$$

The same procedure for $|z| > 3$ leads to

$$f(z) = \sum_{k=2}^{\infty}(3^{k-1}-1)\cdot z^{-k}$$

on R_3. Of course this series converges on $K_{3,\infty}$.

Exercise G3 (On residues of holomorphic functions)

Let $f : \Omega \to \mathbb{C}$ be a holomorphic function and assume there is an $r > 0$ such that $K_{r,0}(z_0) \subseteq \Omega$ where $K_{r,0}(z_0) := \{z \in \mathbb{C} : \ 0 < |z - z_0| < r\}$.
Remember that the *residue of f in z_0* is defined by $\mathrm{Res}(f, z_0) := a_{-1}$ where $\sum_{k=-\infty}^{\infty} a_k \cdot z^k$ is the Laurent series expansion of f converging in $K_{r,0}(z_0)$ to f.

(a) Let $n \in \mathbb{N}$ be a natural number such that $z \to (z - z_0)^n \cdot f(z)$ has a holomorphic extension on $\Omega \cup \{z_0\}$ (e. g. if f has in z_0 a pole of order at most n). Show:

$$\mathrm{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left((z - z_0)^n \cdot f(z)\right).$$

(b) Let $g, h : \Omega \cup \{z_0\} \to \mathbb{C}$ be holomorphic. Assume that h has in z_0 a zero of order 1 and set $f(z) := \frac{g(z)}{h(z)}$. Show:

$$\mathrm{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}.$$

(c) Calculate the following integrals:

$$\text{(i)} \int_{C_1(0)} \frac{e^z}{\sin(z)} dz, \qquad \text{(ii)} \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx, \qquad \int_{C_1(0)} \frac{1}{|z|} dz.$$

Hints for solution:

(a) By assumption f has a Laurent series expansion $f(z) = \sum_{k=-n}^{\infty} a_k \cdot (z - z_0)^k$. This means

$$(z - z_0)^n \cdot f(z) = \sum_{k=0}^{\infty} c_{k-n} \cdot (z - z_0)^k.$$

The right hand side is a power series converging on $K_r(z_0)$. Thus the $(n-1)$-th derivative of this function in z_0 is given by $(n-1)! \cdot a_{-1} = (n-1)! \cdot \mathrm{Res}(f, z_0)$. This proves the claim.

(b) From (a) follows:

$$\begin{aligned} \mathrm{Res}(f, z_0) &= \lim_{z \to z_0} (z - z_0) \cdot f(z) = \lim_{z \to z_0} g(z) \cdot \frac{z - z_0}{h(z) - h(z_0)} \\ &= \frac{g(z_0)}{h'(z_0)}. \end{aligned}$$

(c) Since $\mathrm{Res}\left(\frac{\exp}{\sin}, 0\right) = \frac{\exp(0)}{\cos(0)} = 1$ we get $\int_{C_1(0)} \frac{e^z}{\sin(z)} dz = 2\pi i$.
Since

$$\sum_{z_0 \in \mathbb{C} : \mathrm{Re}(z_0) > 0} \mathrm{Res}\left(z \to \frac{1}{(1+z^2)^2}, z_0\right) = \frac{-i}{4}$$

and $\deg((1+z^2)^2) - \deg(1) \geq 2$ we get

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx = \int_{C_1(i)} \frac{1}{(1+z^2)^2} dz = \frac{\pi}{2}.$$

Since the function $z \to \frac{1}{|z|}$ is not holomorphic, we can't apply integral formulas from complex analysis. But we can calculate the last integral elementary:

$$\int_{C_1(0)} \frac{1}{|z|} dz = \int_{C_1(0)} 1 \, dz = 0.$$

Exercise G4 (Singularities)

If $f : \Omega \to \mathbb{C}$ is holomorphic we call a point $z_0 \in \mathbb{C}$ an *isolated singularity* of f if $z_0 \notin \Omega$ and $K_{r,0}(z_0) = \{z \in \mathbb{C} : \ 0 < |z - z_0| < r\} \subseteq \Omega$ for some $r > 0$. We want to discuss three types of singularities:

An isolated singularity z_0 of f is called a *removable singularity* if f has a holomorphic extension on $\Omega \cup \{z_0\}$.
An isolated singularity z_0 of f is called a *pole* if z_0 is not a removable singularity of f and there exists a $n > 0$ such that $z \to (z - z_0)^n \cdot f(z)$ has a removable singularity in z_0. The smallest number $n \in \mathbb{N}$ with this property is called the *order* of the pole.
An isolated singularity z_0 of f is called an *essential singularity* if z_0 is neither a removable singularity nor a pole.

(a) Find an example for each kind of an isolated singularity.

(b) Show: Let $f : \Omega \to \mathbb{C}$ be holomorphic and z_0 be an isolated singularity. Then there are equivalent:
   (i) The singularity z_0 is removable.
   (ii) There is a power series expansion of f in z_0 converging on $K_r(z_0)$.

(c) Show: Let $f : \Omega \to \mathbb{C}$ be holomorphic and $z_0 \in \Omega$ be an isolated singularity. Then there are equivalent:
   (i) The singularity z_0 is a pole.
   (ii) The principal part of the Laurent series expansion of f in z_0 on $K_{r,0}(z_0)$ is not trivial and all but finitely many coefficients vanish.

(d) Consider the holomorphic functions

$$f(z) = \frac{\sin(z)}{z}, \quad g(z) = \sin\left(\frac{1}{z}\right), \quad h(z) = \frac{1}{\sin(z)}$$

on there natural domains. Each of these functions have in $z_0 = 0$ an isolated singularity. Classify the isolated singularities. **Hint:** You could use the result of (f).

(e) In excercise G2 you determined some Laurent series in z_0 with infinite principal part. Does this mean the function f has an essential singularity in $z_0 = 0$?

(f) Let $f : \Omega \to \mathbb{C}$ be holomorphic and $z_0 \in \Omega$ be a pole of f. Then $\lim_{z \to z_0} |f(z_0)| = \infty$.

(g) The function $f(z) := \exp\left(-\frac{1}{z^2}\right)$ has an essential singularity in $z_0 = 0$. Show: For each $\omega \in \mathbb{C}$ there is a null sequence $(z_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} f(z_n) = \omega$.

The phenomenon in (g) is typical for essential singularities cf. the Casorati-Weierstrass Theorem or – a much stronger fact – the Big Picard Theorem in the literature.

Hints for solution:

(a) Removable: $f(z) := \frac{z}{z}$.
   Pole of order $n \in \mathbb{N}$: $g(z) = \frac{1}{z^n}$.
   Essential singularity: $h(z) := \exp\left(\frac{1}{z^{42}}\right)$.

(b) If f has a removable singularity in z_0 we know f determines a unique holomorphic extension admitting a power series converging on a small open disc around the singularity.
   On the opposite if f has a power series expansion in z_0 then there is a holomorphic extension of f on a small disc around z_0 given by the series.

(c) If z_0 is a pole of order n then $(z-z_0)^n f(z)$ has a removable singularity and thus a power series expansion. Dividing by $(z-z_0)^n$ leads to the Laurent series expansion of f in z_0 and thus the Laurent series has non trivial and finite principal part.
Conversely if $f(z) = \sum_{k=-n}^{\infty} a_k (z-z_0)^k$ then $(z-z_0)^n f(z)$ has a removable singularity in z_0 by (b).

(d) We give the Laurent series expansions of f and g:

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k} \\ g(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{-(2k+1)} \end{aligned}$$

Thus f has a removable singularity in $z_0 = 0$ and g has an essential singularity in $z_0 = 0$. For h we consider

$$H(z) := \frac{z}{\sin(z)}.$$

This function has a removable singularity in $z_0 = 0$ since f has a removable singularity in $z_0 = 0$. This means that h has a pole of order 1 in $z_0 = 0$ or a removable singularity. Since $\lim_{x\in]0,1],\ x\to 0} h(x) = \infty$ we conclude that h has no holomorphic extension in $z_0 = 0$.
Alternatively we can argue that g can't have a removable singularity or a pole at the point $z_0 = 0$: For every $n \in \mathbb{N}$ the function $g_n(z) := z^n \cdot g(z)$ has infinitely many zeroes in every neighbourhood of $z_0 = 0$. If g_n would have a removable singularity the function must be the zero function by the identity theorem. This can't be true.

(e) The expansions in R_2 and R_3 are on annuli which are different from $K_{r,0}(z_0)$. In fact there is no singularity of f in $z_0 = 0$.

(f) Since f has a pole there is a $n \in \mathbb{N}$ such that $(z-z_0)^n f(z)$ is bounded on $K_r(z_0)$ in both directions. This means there are $c, C \in]0,\infty[$ with

$$c \leq |f(z)| \cdot |z-z_0|^n \leq C$$

for all z in a small disc $K_r(z_0)$. From this follows the claim:

$$|f(z)| \geq \frac{c}{|z-z_0|^n}.$$

(g) Fix $\omega \in \mathbb{C} \setminus \{0\}$. Choose a $z \in \mathbb{C}$ with $|z| > 0$, $\mathrm{Im}(z) > 0$ and $e^z = \omega$. Since the exponential map is 2π-periodic we also have $e^{z+2n\pi i} = \omega$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we choose a number $x_n \in \mathbb{C}$ with $x_n^2 = \frac{1}{z+2n\pi i}$ then $\lim_{n\to\infty} x_n = 0$ so we can form the null sequence

$$z_n := -x_n$$

and get

$$\lim_{n\to\infty} f(z_n) = \exp\left((-1)^2 \frac{1}{\frac{1}{z+2n\pi i}} \right) = \exp(z + 2ni\pi) = \omega.$$

For $\omega = 0$ choose an arbitrary real null sequence since there is a real continuous extension of $e^{-\frac{1}{x^2}}$ on the whole real axis.