Analysis III – Complex Analysis
Hints for solution for the
7. Exercise Sheet

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Groupwork

Exercise G1 (The Fundamental Theorem of Algebra)

Use Liouville's Theorem to prove the Fundamental Theorem of Algebra: Every polynomial $p : \mathbb{C} \to \mathbb{C}$ which has no root is constant.
Hint: Consider the rational function $f(z) = \frac{1}{p(z)}$. Show this function has to be bounded if p has no roots.

Hints for solution: Assume $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial without a root in $\mathbb{C}$. Then we have

$$p(z) = z^n \cdot \left(a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n} \right).$$

We can find a $R > 0$ and a $A > 0$ such that $|p(z)| > A$ for $|z| \geq R$. By assumption $\frac{1}{p}$ is holomorphic. Since this function is bounded on $K_R(0)^C$ by the previous calculation and on $K_R(0)$ by compactness we see that $\frac{1}{p}$ is an entire bounded function, i. e. constant.

Exercise G2 (Complex powers of complex numbers)
Let $z, \omega \in \mathbb{C} \setminus \{0\}$ be complex numbers and let $l : \Omega \to \mathbb{C}$ be a logarithm with $z \in \Omega$. We define

$$z^\omega := \exp(l(z) \cdot \omega).$$

Of course this definition depends on the logarithm l. For simplicity we shall choose the principal value Log of the logarithm, i. e. the logarithm function on $\Omega := \mathbb{C} \setminus]-\infty, 0[$ with $\mathrm{Log}(1) = 0$.

(a) Determine i^i.

(b) One might expect the identities

$$\begin{aligned} z^{\omega_1+\omega_2} &= z^{\omega_1} \cdot z^{\omega_2}, \\ z_1^\omega \cdot z_2^\omega &= (z_1 \cdot z_2)^\omega, \\ (z^{\omega_1})^{\omega_2} &= z^{\omega_1 \cdot \omega_2}. \end{aligned}$$

Discuss this.

Hints for solution:

(a) We calculate

$$i^i = \exp(\mathrm{Log}(i) \cdot i) = \exp\left(\frac{\pi}{2} \cdot i^2\right) = \exp\left(-\frac{\pi}{2}\right) \in \mathbb{R}.$$

(b) Only the formula $z^{\omega_1} \cdot z^{\omega_2} = z^{\omega_1+\omega_2}$ holds globally.

Exercise G3 (The complex sine function)

(a) Determine every zero of the complex sine, i. e. every $z \in \mathbb{C}$ with $\sin(z) = 0$.

(b) Show: The function $f(z) := \frac{\sin(z)}{z}$ is holomorphic on $\Omega := \mathbb{C} \setminus \{0\}$ and has a unique holomorphic extension to an entire function.

(c) Determine the integrals

$$\text{(i)} \int_{C_1(0)} \frac{z}{\sin(z)} dz \quad \text{and} \quad \text{(ii)} \int_{C_1(0)} \frac{1}{\sin(z)} dz.$$

Hints for solution:

(a) Let $z \in \mathbb{C}$ be a complex number with $\sin(z) = 0$. Then we get

$$\begin{aligned} & \sin(z) = 0 \\ \Rightarrow\ & \frac{e^{iz} - e^{-iz}}{2i} = 0 \\ \Rightarrow\ & e^{iz} = e^{-iz} \\ \Rightarrow\ & e^{2iz} = 1 \\ \Rightarrow\ & 2iz \in 2\pi i\mathbb{Z} \\ \Rightarrow\ & z \in \pi\mathbb{Z}. \end{aligned}$$

Especially the zeros of sin are real numbers.

(b) Of course $\frac{\sin(z)}{z}$ is holomorphic for all $z \in \mathbb{C} \setminus \{0\}$. Further we can expand sin into a power series and see

$$\frac{\sin(z)}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}.$$

This series converges on $\mathbb{C}$ and this means the left hand side is holomorphic on $\mathbb{C}$.

(c) Since $\frac{\sin(z)}{z}$ has no zeroes in $\overline{\mathbb{D}}$ we get

$$\oint_{C_1(0)} \frac{z}{\sin(z)} dz = 0.$$

With the Cauchy integral formula and $g(z) := \frac{\sin(z)}{z}$ we see

$$\begin{aligned} \oint_{C_1(0)} \frac{1}{\sin(z)} dz &= \oint_{C_1(0)} \frac{\frac{z}{\sin(z)}}{z} dz = \oint_{C_1(0)} \frac{\frac{1}{g(z)}}{z} dz \\ &= 2\pi i \frac{1}{g(0)} = 2\pi i \end{aligned}$$

since $\frac{1}{g}$ is holomorphic on a disc with center 0 and radius smaller than π.

Exercise G4 (Cauchy Integral Formula)

Determine the integrals

$$(i)\ \int_{C_2(i)} \frac{1}{z^2+4}dz, \qquad (ii)\ \int_{C_2(i)} \frac{1}{(z^2+4)^2}dz.$$

Hints for solution:

$$(i)\ \int_{C_2(i)} \frac{1}{z^2+4}dz = \frac{\pi}{2}.$$

For (ii) we use the Cauchy Integral Formula for the derivatives of a holorphic function:

$$\begin{aligned}\int_{C_2(i)} \frac{1}{(z^2+4)^2}dz &= \int_{C_2(i)} \frac{\frac{1}{(z+2i)^2}}{(z-2i)^2}dz \\ &= \int_{C_2(i)} \frac{f(z)}{(z-2i)^2}dz\end{aligned}$$

for $f(z) = \frac{1}{(z+2i)^2}$. We get

$$\int_{C_2(i)} \frac{1}{(z^2+4)^2}dz = 2\pi i \cdot f'(2i) = \frac{\pi}{16}.$$

Homework

Exercise H1 (A generalisation of Liouville's theorem) (1 point)

Let $f : \mathbb{C} \to \mathbb{C}$ holomorphic. Further assume there are constants $a, b \in]0, \infty[$ and a natural number $n \in \mathbb{N}$ with $|f(z)| \leq a \cdot |z|^n + b$ for all $z \in \mathbb{C}$. Show that f is a polynomial with $\deg(f) \leq n$.

Hints for solution: Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ the power series expansion of f. We get from the Cauchy estimation for a choosen $r > 0$:

$$|a_m| \leq r^{-m} \cdot \max_{|z|=r} |f(z)| \leq a \cdot r^{n-m} + b \cdot r^{-m}.$$

In the limit $r \to \infty$ we get for every $m > n$ the identity $a_m = 0$ thus f is a polynomial of degree not higher than n.

Exercise H2 (Power Series) (1 point)

(a) Let $f : \Omega \to \mathbb{C}$ a holomorphic function and $K_r(z_0) \subseteq \Omega$ for some $r > 0$. If f is unbounded on $K_r(z_0)$ then the power series expansion of f in z_0 has radius of convergence r.

(b) Determine the radius of convergence for the power series expansion in $z_0 = 0$ of the following functions

$$(i) \quad f(z) = \frac{1}{z+i}, \qquad (ii) \quad g(z) = \frac{1}{z^2+z+1}, \qquad (iii) \quad g(z) = \frac{1}{\cos(z)}.$$

Hints for solution:

(a) Since $K_r(z_0) \subseteq \Omega$ we know there is a power series expansion $f(z) = \sum_{n \in \mathbb{N}} a_n (z - z_0)^n$. Since f is unbounded on $K_r(z_0)$ there is a sequence $(z_n)_{n \in \mathbb{N}}$ with $|f(z_n)| > n$. Assume the radius of convergence is bigger then r. Then the series converges on $\overline{K_r(z_0)}$. Since this set is compact and since the function $|f|$ is continuous on $K_r(z_0)$ we conclude $|f|$ is bounded a contradiction.

(b) (i) $r = 1$, (ii) $r = 1$, (iii) $r = \frac{\pi}{2}$.

Exercise H3 (The biholomorphic maps of the open unit disk) (1 point)

In this excercise we discuss the biholomorphic transformations of the open unit disk $\mathbb{D}$, i. e. the set

$$\mathrm{Aut}(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{D},\ f \text{ is holomorphic, bijective and its inverse is again holomorphic}\}.$$

Obviously this set forms a subgroup of the group of all bijections of $\mathbb{D}$. We call an element $f \in \mathrm{Aut}(\mathbb{D})$ an *automorphism of* $\mathbb{D}$.
To understand this group, we first prove Schwarz's Lemma. This will help us to determine the automorphisms which leaves the point $0 \in \mathbb{D}$ fix. Then we classify the automorphisms of $\mathbb{D}$.

(a) Prove Schwarz's Lemma: If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic with $f(0) = 0$ then we have for all $z \in \mathbb{D}$ the estimation $|f(z)| \leq |z|$.
Further if there exists a $z_0 \in \mathbb{D}$ with $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$ then $f(z) = \lambda \cdot z$ for some $\lambda \in \mathbb{T}$, i. e. f is a rotation.
Hint: Consider the function $g(z) := \frac{f(z)}{z}$ and use the maximum principle.

(b) Show that every automorphism $f \in \mathrm{Aut}(\mathbb{D})$ with $f(0) = 0$ is a rotation.

(c) Show that every element of the set

$$J := \left\{ f(z) = \frac{az + b}{\overline{b}z + \overline{a}} \,\middle|\, a, b \in \mathbb{C} : |a|^2 - |b|^2 = 1 \right\}$$

is an automorphism of $\mathbb{D}$ and show that J is a subgroup of $\mathrm{Aut}(\mathbb{D})$. Further show

$$J = \left\{ f(z) = e^{i\varphi} \cdot \frac{z - \omega}{\overline{\omega} \cdot z - 1} \,\middle|\, \omega \in \mathbb{D},\ 0 \leq \varphi < 2\pi \right\}.$$

(d) Fix $\omega \in \mathbb{D}$. Find an automorphism $f \in J$ with $f(0) = \omega$.

(e) Prove: If $H \subseteq \mathrm{Aut}(\mathbb{D})$ is a subgroup which satisfies
   (i) for every $z, w \in \mathbb{D}$ there is an automorphism $f \in H$ with $f(z) = w$ (H acts transitively on $\mathbb{D}$),
   (ii) there is a point $z \in \mathbb{D}$ such that $f \in \mathrm{Aut}(\mathbb{D})$ with $f(z) = z$ implies $f \in H$ (H contains the stabiliser of some $z \in \mathbb{D}$),

   then $H = \mathrm{Aut}(\mathbb{D})$.

Conclude

$$\begin{aligned}\mathrm{Aut}(\mathbb{D}) &= \left\{ f(z) = \frac{az + b}{\overline{b}z + \overline{a}} \,\middle|\, a, b \in \mathbb{C} : |a|^2 - |b|^2 = 1 \right\} \\ &= \left\{ f(z) = e^{i\varphi} \cdot \frac{z - \omega}{\overline{\omega} \cdot z - 1} \,\middle|\, \omega \in \mathbb{D},\ 0 \leq \varphi < 2\pi \right\}.\end{aligned}$$

(f) Show: Every $f \in \mathrm{Aut}(\mathbb{D})$ extends to $\overline{\mathbb{D}}$ and maps $\mathbb{T}$ bijective to $\mathbb{T}$.

Hints for solution:

(a) We have $f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=1}^{\infty} a_n z^n = z \cdot \sum_{n=0}^{\infty} a_{n+1} z^n = z \cdot g(z)$ and $g(0) = a_1 = f'(0)$. From $|f(z)| \leq 1$ we conclude $r \cdot \max_{|z|=r} |g(z)| \leq 1$ for all $0 < r < 1$, i. e. $f(z) \leq |z|$. Further $|f'(0)| = |g(0)| \leq 1$.
Assume $|f'(0)| = 1$ or $|g(c)| = |c|$ for some $c \in \mathbb{D} \setminus \{0\}$. Then we have $|g(0)| = 1$ or $|g(c)| = 1$ which means that g takes a maximum on $\mathbb{D}$. From the maximum principle we follow that g is constant. Thus $f(z) = z \cdot g(z) = z \cdot \lambda$.

(b) Since f is an automorphism the inverse map f^{-1} is again an automorphism and we follow

$$|f(z)| \leq |z| \quad \text{and} \quad |z| = |f^{-1}(f(z)))| \leq |f(z)|$$

for all $z \in \mathbb{D}$. This means $|f(z)| = |z|$ and by (a) $f(z) = \lambda \cdot z$.

(c) Simple calculation.

(d) Take for example

$$f(z) := \frac{z - \omega}{\overline{\omega} \cdot z - 1}.$$

Then we have $f(0) = \omega$.

(e) Let $h \in \mathrm{Aut}(\mathbb{D})$ arbitrary. Since H acts transitively we find an $g \in H$ with $g(h(z)) = z$. Thus $g \circ h$ is an element of the stabiliser of $z \in \mathbb{D}$ and we conclude $g \circ h \in H$. Since H is a subgroup we have

$$h = g^{-1} \circ g \circ h \in H.$$

Thus $H = \mathrm{Aut}(\mathbb{D})$.

(f) Choose an $f \in \mathrm{Aut}(\mathbb{D})$ with $\varphi = 0$. Since $|\omega| < 1$ there is no singularity of f in the set $\overline{\mathbb{D}}$. Thus $f : \overline{\mathbb{D}} \to \mathbb{C}$ is well defined.
For $|z| = 1$ we see

$$\begin{aligned} |f(z)|^2 &= \left|\frac{z - \omega}{\overline{\omega} \cdot z - 1}\right|^2 = \frac{|z|^2 - w\overline{z} - \overline{\omega}z + |\omega|^2}{|\omega|^2|z|^2 - \omega\overline{z} - \overline{\omega}z + 1} \\ &= \frac{1 - w\overline{z} - \overline{\omega}z + |\omega|^2}{|\omega|^2 - \omega\overline{z} - \overline{\omega}z + 1} = 1. \end{aligned}$$

Thus $f(\mathbb{T}) \subseteq \mathbb{T}$.
Choose $\lambda \in \mathbb{T}$ arbitrary and put $z = \frac{\omega - \lambda}{1 - \lambda\overline{\omega}}$. Then we have $|z| = 1$ and $f(z) = \lambda$. This means $f(\mathbb{T}) = \mathbb{T}$. A simple calculation shows that f is injective.
For $\varphi \neq 0$ we know $g(z) := e^{i\varphi} z$ is a bijection of $\mathbb{D}$ and of $\mathbb{T}$. Thus the argumentation above holds for arbitrary automorphisms $f \in \mathrm{Aut}(\mathbb{D})$.