Analysis III – Complex Analysis
Hints for solution for the
6. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

Department of Mathematics
Prof. Dr. Burkhard Kümmerer
Andreas Gärtner
Walter Reußwig

WS 11/12
January 10, 2012

Groupwork

Exercise G1 (Cauchy Integral Formula)

Use the Cauchy Integral Formula to determine the following integrals:

$$(a)\ \oint_{C_1(i)} \frac{1}{z-i}dz, \qquad (b)\ \oint_{C_1(i)} \frac{1}{z^2+1}dz, \qquad (c)\ \oint_{C_{42}(i)} \frac{1}{z^2+1}dz.$$

Hint: To decompose the integral in (c) into elementary circle integrals use the homotopy invariance of the path integral.

Hints for solution: The Cauchy integral formula is usefull to calculate path integrals around singularities of holomorphic functions: (i)

$$\oint_{C_1(i)} \frac{1}{z-i}dz = \oint_{C_1(i)} \frac{f(z)}{z-i}dz = 2\pi i f(i).$$

with $f(z) := 1$ for all $z \in \mathbb{C}$, because this function f is holomorphic on $\overline{K_1(i)}$. Thus

$$\oint_{C_1(i)} \frac{1}{z-i}dz = 2\pi i.$$

(ii) We see with $f(z) = \frac{1}{z+i}$:

$$\begin{aligned} \oint_{C_1(i)} \frac{1}{z^2+1}dz &= \oint_{C_1(i)} \frac{\frac{1}{z+i}}{z-i}dz = \oint_{C_1(i)} \frac{f(z)}{z-i}dz \\ &= 2\pi i \cdot f(i) = \frac{2\pi i}{2i} = \pi. \end{aligned}$$

(iii) Since $C_{42}(0)$ is homotopic in $\mathbb{C} \setminus \{-i, i\}$ in some sense to a path behaving like $C_1(i) + C_1(-i)$ we have to calculate

$$\oint_{C_1(-i)} \frac{\frac{1}{z-i}}{z+i}dz = 2\pi i f(-i) = \frac{2\pi i}{-2i} = -\pi.$$

Thus the integral in (iii) vanishes, since $\pi + (-\pi) = 0$.

Exercise G2 (Radius of convergence)

Consider a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ which has radius of convergence $r > 0$. Show: There is no holomorphic extension of f on $K_R(0)$ for any $R > r$.

Hints for solution: Assume there is a holomorphic extension g of f on a greater disc. Since g is holomorphic we could represent g by a power series with center 0 and radius of convergence R. Since the f and g coincidences on $K_r(0)$ we conclude both power series coincidences. But this means R must be the radius of convergence of f, a contradiction.

Exercise G3 (The Cauchy transform)
Let $f : \mathbb{T} \to \mathbb{C}$ be a continuous function. We define a function $\widehat{f} : \mathbb{D} \to \mathbb{C}$ by

$$\widehat{f}(z) := \frac{1}{2\pi i} \oint_{\mathbb{T}} \frac{f(\omega)}{\omega - z} d\omega.$$

(a) Show that $\widehat{f}$ is holomorphic.

(b) Show: If $f : \Omega \to \mathbb{C}$ with $\overline{\mathbb{D}} \subseteq \Omega$ is holomorphic then $\widehat{f} = f$.

(c) Is it always true that $\lim_{z \to \omega,\, z \in \mathbb{D}} \widehat{f}(z) = f(\omega)$ holds for $\omega \in \mathbb{T}$?

The function $\widehat{f}$ is called the *Cauchy transform* of f.

Hints for solution:

(a) We follow the proof of the main theorem of holomorphy from the lectures and show that g can be represented on $\mathbb{D}$ by a power series. Indeed for $z \in \mathbb{D}$ and $\omega \in \mathbb{T}$ we have $|z| < |\omega| = 1$ and

$$\begin{aligned} \frac{1}{\omega - z} &= \frac{1}{\omega} \cdot \frac{1}{1 - \frac{z}{\omega}} \\ &= \frac{1}{\omega} \cdot \sum_{n=0}^{\infty} \left(\frac{z}{\omega} \right)^n \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{\omega} \right)^{n+1} \cdot z^n. \end{aligned}$$

Further let $M := \|f\|_{\infty, \mathbb{T}}$. Since f is bounded on the circle we get

$$\left| \frac{f(\omega)}{\omega^{n+1}} z^n \right| \leq M \cdot z^n.$$

This shows that

$$\frac{f(\omega)}{\omega - z} = \sum_{n=0}^{\infty} \frac{f(\omega)}{\omega^{n+1}} z^n$$

converges locally uniformly on $\mathbb{D}$. Integrating both sides brings

$$\begin{aligned} g(z) &= \frac{1}{2\pi i} \oint_{\mathbb{T}} \frac{f(\omega)}{\omega - z} d\omega \\ &= \frac{1}{2\pi i} \cdot \oint_{\mathbb{T}} \sum_{n=0}^{\infty} \frac{f(\omega)}{\omega^{n+1}} z^n d\omega \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \cdot \oint_{\mathbb{T}} \frac{f(\omega)}{\omega^{n+1}} d\omega \right) z^n. \end{aligned}$$

We see that g is holomorphic and we have a formula to determine the coefficients of its power series.

(b) This is exactly the Cauchy integral formula.

(c) We give a counterexample. Let $f : \mathbb{T} \to \mathbb{T},\ f(z) := \overline{z} = z^{-1}$. Then we get

$$\begin{aligned} g(z) &= \frac{1}{2\pi i}\oint_{\mathbb{T}} \frac{f(\omega)}{\omega - z} d\omega \\ &= \frac{1}{2\pi i}\oint_{\mathbb{T}} \frac{1}{\omega(\omega - z)} d\omega \\ &= \frac{1}{2\pi i}\oint_{C_r(0)} \frac{1}{\omega(\omega - z)} d\omega + \frac{1}{2\pi i}\oint_{C_R(z)} \frac{1}{\omega(\omega - z)} d\omega \\ &= 2\pi i \cdot \left(-\frac{1}{z} + \frac{1}{z}\right) = 0 \end{aligned}$$

where r and R are small enough. Thus $g(z) = 0$ for all $z \in \mathbb{D}$ and so we see

$$f(1) = 1 \neq 0 = \lim_{z \to 0,\ z \in \mathbb{D}} g(z).$$

This contradicts the conjecture.

Homework

Exercise H1 (Conjugation with reflections) (1 point)

Let $L \subseteq \mathbb{C}$ a one dimensional real subspace of $\mathbb{C}$. Further let $\sigma : \mathbb{C} \to \mathbb{C}$ the real linear map with $\sigma(l) = l$ for all $l \in L$ and $\sigma(l^\perp) = -l^\perp$ for all $l \in L^\perp$ where $L^\perp$ is the orthogonal complement of L with respect to the canonical scalar product on $\mathbb{R}^2$. Of course this means that σ is orthogonal with determinant -1.

(a) Let $\Omega \subseteq \mathbb{C}$ be a domain and $f : \Omega \to \mathbb{C}$ a function. Show that the following statements are equivalent:
  (i) $f : \Omega \to \mathbb{C}$ is holomorphic.
  (ii) $\sigma \circ f \circ \sigma : \sigma(\Omega) \to \mathbb{C}$ is holomorphic.

Now let $\Omega \subseteq \mathbb{C}$ be a domain which is symmetric a to the real axis, i. e. $\Omega = \{\overline{z} : z \in \Omega\}$. We define $f^*(z) := \overline{f(\overline{z})}$. From (a) it follows that f^* is holomorphic if and only if f is holomorphic.

(b) Determine the derivative of f^* directly.

(c) Assume $f(z) = \sum_{k=0}^{\infty} a_k \cdot z^k$ converges on Ω. Determine the power series of f^*. Which holomorphic functions of this form satisfy $f = f^*$?

(d) Show that every holomorphic function on Ω is linear combination of two holomorphic functions g, h on Ω with $g = g^*$ and $h = h^*$.
Hint: To get an idea you could first prove (d) for holomorphic functions given by a power series like in (c).

Hints for solution:

(a) We show (i) $\Rightarrow$ (ii): Since σ is a real linear map the function $\sigma \circ f \circ \sigma$ is real differentiable. The differential is given by

$$\begin{aligned} d(\sigma \circ f \circ \sigma)(x,y) &= d\sigma(f(\sigma(x,y))) \cdot df(\sigma(x,y)) \cdot d\sigma(y) \\ &= \sigma \cdot df(\sigma(x,y)) \cdot \sigma. \end{aligned}$$

We show that the Cauchy-Riemann differential equations are satisfied. Since f is holomorphic by assumption we have for an arbitrary but fixed $(x,y) \in \mathbb{R}^2$:

$$df(\sigma(x,y)) = \lambda \cdot \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \lambda \cdot T$$

where the matrix T is an orthogonal matrix with determinant 1. Thus

$$df(x,y) = \lambda \cdot (\sigma \cdot T \cdot \sigma) = \lambda \cdot S$$

and S is a matrix with determinant 1 and a product of three orthogonal matrices. Thus S is again an orthogonal matrix. This means $df(x,y)$ satisfies the Cauchy-Riemann differential equations.
(ii) $\Rightarrow$ (i): We know that if f is holomorphic so $\sigma \circ f \circ \sigma$ is holomorphic, too. Set $g(z) := \sigma \circ f \circ \sigma(z)$. By assumption g is holomorphic. We know by the first step that $\sigma \circ g \circ \sigma$ is holomorphic, too. Thus

$$\sigma \circ g \circ \sigma = \sigma^2 \circ f \circ \sigma^2 = f.$$

This means f is holomorphic.

(b) We get

$$
\begin{aligned}
(f^*)'(z) &= \lim_{z\to z_0} \frac{f^*(z) - f^*(z_0)}{z - z_0} \\
&= \lim_{z\to z_0} \frac{\overline{f(\overline{z})} - \overline{f(\overline{z_0})}}{z - z_0} \\
&= \lim_{z\to z_0} \overline{\left(\frac{f(\overline{z}) - f(\overline{z_0})}{\overline{z} - \overline{z_0}}\right)} \\
&= \overline{\left(\lim_{z\to z_0} \frac{f(\overline{z}) - f(\overline{z_0})}{\overline{z} - \overline{z_0}}\right)} \\
&= \overline{f'(\overline{z_0})}.
\end{aligned}
$$

(c) Of course $f^*(z) = \sum_{k=0}^{\infty} \overline{a_k} \cdot z^k$. Thus we have $f = f^*$ if and only if all coefficients of the power series are real numbers.

(d) Set $g = \frac{1}{2}(f + f^*)$ and $h := \frac{i}{2} \cdot (f - f^*)$. Then we have $f = g - ih$.

Exercise H2 (The mean value property) (1 point)

Let $\Omega \subseteq \mathbb{C}$ a simply connected domain.

(a) Show that the following statements are equivalent for a function $u : \Omega \to \mathbb{R}$:

(i) $\Delta u(z) = 0$ for each $z \in \Omega$ where Δ is the Laplacian if we identify Ω as a subset of $\mathbb{R}^2$.

(ii) There is a holomorphic function $f : \Omega \to \mathbb{C}$ with $u(z) = \mathrm{Re}(f(z))$.

We call a function u satisfiing (i) *harmonic on* Ω.

(b) Show the mean value property for harmonic functions: If $u : \Omega \to \mathbb{R}$ is harmonic and $\overline{K_r(z_0)} \subseteq \Omega$ holds, then

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + r \cdot e^{it}) dt.$$

Now let $\Omega \subseteq \mathbb{C}$ be an arbitrary domain.

(c) Let $u : \Omega \to \mathbb{R}$ be a harmonic function. Since Ω need not to be simply connected, we can't conclude that u is the real part of a holomorphic function. Why does u satisfy the mean value property anyway?

Hints for solution:

(a-) Using the Cauchy Integral Formula we get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{C_r(z_0)} \frac{f(\omega)}{\omega - z} d\omega \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + r \cdot e^{it})}{z_0 - z + r \cdot e^{it}} \cdot ir \cdot e^{it} dt. \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(z_0 + r \cdot e^{it})}{z_0 - z + r \cdot e^{it}} \cdot r \cdot e^{it} dt. \end{aligned}$$

This equation holds for $z = z_0$ and we see

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt.$$

(a) If $\Delta u = 0$ we have $\partial_x^2 u = -\partial_y^2 u$. Thus the vector field $(-\partial_y u, \partial_x u)$ satisfies the integrability conditions. This means there is a potential $v : \Omega \to \mathbb{R}$ for u with

$$\partial_x v = -\partial_y u, \qquad \partial_y v = \partial_x u.$$

Further v is harmonic and $f(x + iy) := u(x, y) + iv(x, y)$ satisfies the Cauchy-Riemann differential equations by construction. This means u is the real part of a holomorphic function. In opposite a real part of a holomorphic function is harmonic since the Cauchy-Riemann differential equations are satisfied.

(b) Integrate only the real part in (a-) and you get the mean value property for harmonic functions:

$$\mathrm{Re}(f(z_0)) = \mathrm{Re}\left(\frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r \cdot e^{it}) dt \right) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}(f(z_0 + r \cdot e^{it})) dt.$$

(c) Yes since $\overline{K_r(z_0)} \subseteq \Omega$ means that there is a simply connected open subset $U \subseteq \Omega$ with $\overline{K_r(z_0)} \subseteq U$ and u is harmonic on U. This implies u is locally the real part of a holomorphic function and we can apply the Cauchy integral formula and everything we had done stays locally true. Especially the mean value property.

Exercise H3 (Real integrals and complex path integrals) (1 point)

In this exercise we want to calculate the integral:

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx.$$

We will see that the complex line integral could be a mighty help for real integration.

(a) Calculate the roots of the polynomial $p(z) = z^4 + 1$. Sketch them into the unit circle and decide which of them lie in the upper half plane $H := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

(b) Show that the real integral

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx$$

exists and is finite.

(c) Let $r \in]1, \infty[$ an arbitrary number. Consider the paths

$$\begin{aligned} \gamma_r^{(1)} : [0,1] \to \mathbb{C}, \quad \gamma_r^{(1)}(t) &:= r(2t-1), \\ \gamma_r^{(2)} : [0,1] \to \mathbb{C}, \quad \gamma_r^{(2)}(t) &:= r \cdot e^{i\pi t}. \end{aligned}$$

and set $\gamma_r := \gamma_r^{(1)} + \gamma_r^{(2)}$. Assure yourself that γ_r is a loop in $\mathbb{C}$. Sketch the path γ_r for a suitable choice of $r > 1$ and argue that

$$\int_{\gamma_r} \frac{z^2}{z^4+1} dz$$

is independent of the choosen $r \in]1, \infty[$.

(d) Use the standard estimation to show

$$\lim_{r \to \infty} \int_{\gamma_r^{(2)}} \frac{z^2}{z^4+1} dz = 0. \quad \text{Conclude:} \quad \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx = \int_{\gamma_r} \frac{z^2}{z^4+1} dz$$

for any $r > 1$.

(e) Use the Cauchy Integral Formular and the factorisation of $z^4 + 1$ into linear factors to show

$$\frac{1}{2\pi i} \cdot \oint_{C_1(\xi_1)} \frac{z^2}{z^4+1} dz = \frac{\xi_1^2}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)(\xi_1 - \xi_4)}$$

$$\frac{1}{2\pi i} \cdot \oint_{C_1(\xi_2)} \frac{z^2}{z^4+1} dz = \frac{\xi_2^2}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)(\xi_2 - \xi_4)}$$

where $\xi_k = e^{\frac{\pi i (2k+1)}{4}}$ are the roots of $z^4 + 1$.

(f) Argue

$$\int_{\gamma_r} \frac{z^2}{z^4+1} dz = \oint_{C_1(\xi_1)} \frac{z^2}{z^4+1} dz + \oint_{C_1(\xi_2)} \frac{z^2}{z^4+1} dz$$

for any $r > 1$. Finally determine this integral.

Hints for solution:

(a) Assume $\xi \in \mathbb{C}$ is a root of $X^4 + 1$. Then we have $\xi^4 = -1$ and this means

$$\xi \in \left\{ e^{\frac{i\cdot\pi}{4}\cdot(2k+1)} : 0 \leq k \leq 3 \right\} =: N.$$

(b) We estimate the real integral:

$$\begin{aligned}
\left| \int_{-\infty}^{\infty} \frac{x^2}{1+x^4} \right| dx &= \int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx \\
&= \int_{-1}^{1} \frac{x^2}{1+x^4} dx + 2 \cdot \int_{1}^{\infty} \frac{x^2}{1+x^4} dx \\
&\leq \int_{-1}^{1} 1 \cdot dx + 2 \cdot \int_{1}^{\infty} \frac{x^2}{1+x^4} dx \\
&\leq 2 + 2 \cdot \int_{1}^{\infty} \frac{1}{x^2} dx \\
&= 2 + \frac{2}{3} < \infty.
\end{aligned}$$

This means the real integral exists.

(c) Since γ_r is homotopic to γ_s for $1 < r, s$ we get the claim by the cauchy integral theorem or the real path integral. Recognize, the function f is defined on $\mathbb{C} \setminus N$. So all holes/singularities occur in a circle of radius 1.

(d) We use $|\gamma(z)| = r$ and $L(\gamma_r^{(2)}) = r \cdot \pi$ and get

$$\begin{aligned}
\int_{\gamma_r^{(2)}} f(z) dz &\leq \max_{\{|z|=r,\ \mathrm{Im}(z)>0, r>1\}} |f(z)| \cdot L(\gamma_r^{(2)}) \\
&= \max_{\{|z|=r,\ \mathrm{Im}(z)>0, r>1\}} |\frac{|z|^2}{1+|z|^4} \cdot \pi \cdot r \\
&\leq \frac{r \cdot \pi}{r^2} = \frac{\pi}{r}.
\end{aligned}$$

So we get the result by taking the limit $r \to \infty$. The conclusion is clear since the path integral is additive.

(e) We have the factorisation

$$f(z) = \frac{z^2}{(z-\xi_1)(z-\xi_2)(z-\xi_3)(z-\xi_4)}.$$

Define the functions

$$g(z) = \frac{z^2}{(z-\xi_2)(z-\xi_3)(z-\xi_4)},$$

$$h(z) = \frac{z^2}{(z-\xi_1)(z-\xi_3)(z-\xi_4)}.$$

Then g is holomorphic on $K_1(\xi_1)$ and h is holomorphic on $K_1(\xi_2)$. We get by the Cauchy Integral Formular:

$$\begin{aligned} \frac{1}{2\pi \cdot i} \oint_{K_1(\xi_1)} f(z) dz &= \frac{1}{2\pi \cdot i} \oint_{K_1(\xi_1)} \frac{\left(\frac{z^2}{(z-\xi_2)(z-\xi_3)(z-\xi_4)}\right)}{z-\xi_1} dz. \\ &= g(\xi_1) = \frac{\xi_1^2}{(\xi_1-\xi_2)(\xi_1-\xi_3)(\xi_1-\xi_4)}. \end{aligned}$$

The last equation uses the holomorphy of g on $K_1(\xi_1)$. Analogously we can calculate the other path integral.

(f) We can deform the two paths $\gamma_1(t) = \xi_1 + e^{2\pi i t}$ and $\gamma_2(t) = \xi_2 + e^{2\pi i t}$ to a loop γ surrounding ξ_1 and ξ_2 once with $\oint_\gamma f\, dz = \oint_{\gamma_1} f\, dz + \oint_{\gamma_2} f\, dz$. This path is then homotopic to γ_r. We can finally determine this path integral:

$$\begin{aligned} \xi_1 - \xi_2 &= \sqrt{2} \\ \xi_1 - \xi_3 &= \sqrt{2} \cdot (1+i) \\ \xi_1 - \xi_4 &= \sqrt{2} \cdot i \\ \xi_2 - \xi_1 &= -\sqrt{2} \\ \xi_2 - \xi_3 &= \sqrt{2} \cdot i \\ \xi_2 - \xi_4 &= \sqrt{2}(-1+i) \\ \xi_1^2 &= i \\ \xi_2^2 &= -i \\ \oint_{\gamma_r} f(z) dz &= 2\pi \cdot i \cdot \big(g(\xi_1) + h(\xi_2)\big) = 2\pi \cdot i \cdot \left(\frac{-1-i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}} \right) \\ &= \frac{\pi}{\sqrt{2}}. \end{aligned}$$