Analysis III – Complex Analysis
Hints for solution for the
5. Exercise Sheet

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Groupwork

Exercise G1 (Standard estimations)

(a) Consider a continuous function $f : [0,1] \to \mathbb{C}$. Show

$$\left|\int_0^1 f(t)dt\right| \le \int_0^1 \left|f(t)\right|dt.$$

Hint: For each complex number $z \in \mathbb{C}$ there is a complex number $\omega \in \mathbb{C}$ such that $\omega \cdot z = |z|$.

(b) Let $\Omega \subseteq \mathbb{C}$ be a domain and $f : \Omega \to \mathbb{C}$ be a continuous function. Further let $\gamma : [0,1] \to \Omega$ be an arbitrary path. Show the standard estimation for the complex path integral:

$$\left|\int_\gamma f(z)dz\right| \le \sup_{t\in[0,1]} \{|f(\gamma(t))|\} \cdot L(\gamma) < \infty.$$

Hints for solution: Since the integral is a complex number we can find a complex number $c \in \mathbb{T}$ (with $|c| = 1$) such that

$$c \cdot \int_0^1 f(t)dt \in \mathbb{R}.$$

Now a simple calculation shows

$$\begin{aligned} c \cdot \int_0^1 f(t)dt &= \operatorname{Re}\left(\int_0^1 c \cdot f(t)dt\right) = \operatorname{Re}\left(\int_0^1 \operatorname{Re}(c \cdot f(t))dt + i\int_0^1 \operatorname{Im}(c \cdot f(t))dt\right) \\ &= \int_0^1 \operatorname{Re}(c \cdot f(t))dt. \end{aligned}$$

This means

$$\begin{aligned} \left|\int_0^1 f(t)dt\right| &= \left|c \cdot \int_0^1 f(t)dt\right| = \left|\int_0^1 \operatorname{Re}(c \cdot f(t))dt\right| \\ &\le \int_0^1 \left|\operatorname{Re}(c \cdot f(t))\right|dt \le \int_0^1 \left|c \cdot f(t)\right|dt \\ &= \int_0^1 \left|f(t)\right|dt. \end{aligned}$$

A simple calculation shows:

$$\begin{aligned}\left|\int_\gamma f(z)dz\right| &= \left|\int_0^1 f(\gamma(t))\gamma'(t)dt\right| \\ &\leq \int_0^1 |f(\gamma(t))|\cdot|\gamma'(t)|dt \\ &\leq \int_0^1 \sup_{t\in[0,1]}\{|f(\gamma(t))|\}\cdot|\gamma'|dt \\ &= \sup_{t\in[0,1]}\{|f(\gamma(t))|\}\cdot L(\gamma).\end{aligned}$$

Since $[0,1]$ is compact and f is continuous on the curve of γ we get that $\sup_{t\in[0,1]}\{|f(\gamma(t))|\}$ exists and is finite.

Exercise G2 (Locally uniform convergence)

Let $\Omega \subseteq \mathbb{C}$ be a domain. If $f : \Omega \to \mathbb{C}$ is an arbitrary function then we denote by f^K for $K \subseteq \Omega$ the restriction of f on K, i. e. $f^K : K \to \mathbb{C}$, $f^K(z) := f(z)$.
Let $f_n : \Omega \to \mathbb{C}$ be a function for each $n \in \mathbb{N}$. We say the sequence $(f_n)_{n\in\mathbb{N}}$ converges **locally uniformly** to a function $f : \Omega \to \mathbb{C}$ if for each compact subset $K \subseteq \Omega$ the sequence $(f_n^K)_{n\in\mathbb{N}}$ converges uniformly to f^K.

(a) Show: If $(f_n)_{n\in\mathbb{N}}$ converges locally uniformly to f then it converges pointwise to f.

(b) Show: If $(f_n)_{n\in\mathbb{N}}$ converges locally uniformly to f and if the function f_n is continuous for each $n \in \mathbb{N}$ then f is continuous.

(c) Give an example of a locally uniformly convergent sequence which is not uniformly convergent.

(d) Show: If $(f_n)_{n\in\mathbb{N}}$ converges locally uniformly to f then for every path $\gamma : [0,1] \to \Omega$ we have

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz.$$

(e) Consider the domain $\Omega := \mathbb{C} \setminus \{0\}$ and the following rational functions:

$$f_n(z) := \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{1}{z^k}.$$

Show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges locally uniformly to $f(z) = e^{\frac{1}{z}}$ and determine the path integral $\oint_{K_1(0)} e^{\frac{1}{z}} dz$.

Hints for solution:

(a) Since for every $z \in \Omega$ the set $\{z\}$ is compact we get pointwise convergence.

(b) Since Ω is open, every point $z \in \Omega$ has a compact neighbourhood: Take a $z \in \Omega$. Since Ω is open, there is an $\varepsilon > 0$ such that $K_\varepsilon(z) \subseteq \Omega$. This means $K(z) := \overline{K_{\frac{\varepsilon}{42}}(z)} \subset K_\varepsilon(z) \subseteq \Omega$. Since $K(z)$ is a neighbourhood, every sequence in Ω which converges to z has all but finitely many elements of the sequence belonging to $K(z)$. Thus f is continuous in z if $f^{K(z)}$ is continuous in z.
Since the sequence converges uniformly on each compact subset of Ω to f, we have that $f^{K(z)}$ is continuous as uniform limit of continuous functions. Thus f is continuous in $z \in \Omega$ and since z was arbitrary we have f is continuous on Ω.

(c) The function sequence $f_n(z) := \sum_{k=0}^{n} z^k$ converges locally uniformly to $\frac{1}{1-z}$ on $\mathbb{D}$ but not uniformly since the limit function is unbounded and so every distance $\left\|f - f_n\right\|_\infty$.

(d) We set $\Gamma := \gamma([0,1])$. This set is compact. If $\varepsilon > 0$ is given we find a n_0 such that for every $n > n_0$ we have

$$|f_n(z) - f(z)| \le \frac{\varepsilon}{L(\gamma)}.$$

Using the standard estimation we get:

$$\begin{aligned}\left|\int_\Gamma f_n(z)dz - \int_\Gamma f(z)dz\right| &\leq \int_\Gamma |f_n(z) - f(z)|dz \\ &\leq \frac{\varepsilon}{L(\gamma)} \cdot L(\gamma) = \varepsilon.\end{aligned}$$

This means of course $\lim_{n\to\infty} \int_\Gamma f_n(z)dz = \int_\Gamma f(z)dz$.

(e) Let $K \subseteq \Omega$ compact. Then $K' := \{\frac{1}{z} : z \in K\}$ is compact, too. Further $\sum_{k=0}^{n} \frac{1}{k!} z^k$ converges locally uniformly to the exponential function (cf. Analysis II). Putting this together we get the claim.
Using (d) and using

$$\oint_{K_1(0)} \sum_{k=0}^{n} \frac{1}{k!} z^{-k} dz = 2\pi \cdot i \cdot \frac{1}{1!} = 2\pi \cdot i,$$

we get

$$\oint_{K_1(0)} f(z)dz = 2\pi \cdot i.$$

Exercise G3 (Radius of convergence)

Let $(a_n)_{n\in\mathbb{N}}$ be a monotonically decreasing null sequence and define the power series

$$f(z) := \sum_{n=0}^{\infty} a_n z^n.$$

(a) Show that the radius of convergence of f is at least 1.

(b) Show that for each $z \in \mathbb{T} \setminus \{1\} = \{z \in \mathbb{C} \setminus \{1\} : |z| = 1\}$ the series converges.

Hint: Use $\sum_{k=m}^{n} a_k z^k = \frac{1}{1-z} \cdot \left((1-z) \cdot \sum_{k=m}^{n} a_k z^k\right)$ and estimate the second term.

Hints for solution:

(a) Clear.

(b)

$$\begin{aligned} \left|\sum_{k=m}^{n} a_k z^k\right| &= \left|\frac{1}{1-z}\right| \left|(1-z) \cdot \sum_{k=m}^{n} a_k z^k\right| \\ &\leq \left|\frac{2|a_m|}{1-z}\right|. \end{aligned}$$

The last term is a Cauchy sequence so we get the claim.

Homework

Exercise H1 (Real parts of complex differentiable functions) (1 point)

Consider the polynomial $p(x,y) := x^2 + 2axy + by^2$ where $a, b \in \mathbb{R}$ are parameters. Decide for which choices $a, b \in \mathbb{R}$ the polynomial is the real part of a complex differentiable function $f : \mathbb{C} \to \mathbb{C}$, i. e.

$$p(x,y) = \mathrm{Re} f(x+iy).$$

On the other side if $p(x,y) = \mathrm{Re} f(x+iy) = \mathrm{Re} g(x+iy)$ for complex differentiable functions $f, g : \mathbb{C} \to \mathbb{C}$ what can you say about the relationship of f and g?

Hints for solution: We use the Cauchy Riemann Differential Equations and get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We calculate

$$\frac{\partial u}{\partial x} = 2x + 2ay \quad \frac{\partial u}{\partial y} = 2ax + 2by.$$

Since the integrability conditions have to be satisfied we get

$$\frac{\partial^2 v}{\partial x \partial y} = 2 = -2b = \frac{\partial^2 v}{\partial y \partial x}$$

and conclude $b = -1$ and a could be arbitrary. This is true because $v : \mathbb{R}^2 \to \mathbb{R}$ is a potential for $(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x})^T$.
Taking $\gamma(t) := (x,y) \cdot t$ with $\gamma : [0,1] \to \mathbb{C}$ and calculating the path integral we get

$$v(x,y) = a(y^2 - x^2) + 2xy + C.$$

We have

$$f(x+iy) = x^2 + 2axy - y^2 + i \cdot (a(y^2 - x^2 + 2xy + C).$$

Of course the difference of two such functions must be strictly imaginary and constant.

Exercise H2 (The Gamma function) (1 point)

There are many connections between number theory and complex analysis. In this excercise we construct a complex differentiable function $\Gamma : \Omega \to \mathbb{C}$ which interpolates the factorials. We use the following result:

Theorem: Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f : \Omega \times]0, \infty[\to \mathbb{C}$ a function satisfying the following three conditions:

(i) For every $z \in \Omega$ one has $\int_0^\infty |f(z,t)| dt < \infty$.

(ii) For every $t \in]0, \infty[$ the function $z \to f(z,t)$ is complex differentiable.

(iii) For every compact disk $K = K_r(z_0) \subseteq \Omega$ there is a positive function $g_K :]0, \infty[\to \mathbb{R}_0^+$ with $|f(z,t)| \leq g_K(t)$ for all $t \in]0, \infty[$ and all $z \in K$ and one has

$$\int_0^\infty g_K(t) dt < \infty.$$

Then the function $F : \Omega \to \mathbb{C}$,

$$F(z) := \int_0^\infty f(z,t) dt$$

is complex differentiable and its derivatives are given by

$$F^{(n)}(z) = \int_0^\infty \frac{\partial^n}{\partial z^n} f(z,t) dt.$$

Now we use the domain $\Omega_+ := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ which is the open right half complex plane.

(a) Show that the following statements are equivalent for $\Omega = \Omega_+$:

(iii) For every compact disk $K = K_r(z_0) \subseteq \Omega_+$ there is a positive function $g_K :]0, \infty[\to \mathbb{R}_0^+$ with $|f(z,t)| \leq g_K(t)$ for all $t \in]0, \infty[$ and all $z \in K$ and one has

$$\int_0^\infty g_K(t) dt < \infty.$$

(iii') For every compact rectangular $K = [a,b] \times i \cdot [c,d] \subseteq \Omega_+$ there is a positive function $g_K :]0, \infty[\to \mathbb{R}_0^+$ with $|f(z,t)| \leq g_K(t)$ for all $t \in]0, \infty[$ and all $z \in K$ and one has

$$\int_0^\infty g_K(t) dt < \infty.$$

(b) Show that

$$\Gamma_+(z) := \int_0^\infty t^{(z-1)} \cdot e^{-t} dt := \int_0^\infty e^{\ln(t)\cdot(z-1)} \cdot e^{-t} dt$$

defines a complex differentiable function on Ω_+.

(c) Show the following formulas:

$$\begin{aligned} \Gamma_+(1) &= 1, \\ \Gamma_+(z+1) &= z \cdot \Gamma_+(z) \quad \text{for all } z \in \Omega_+. \end{aligned}$$

Conclude $\Gamma_+(n+1) = n!$ which means the function Γ_+ is indeed a complex differentiable interpolation of the factorials on the right complex half plane.

(d) Show that the function Γ_+ is bounded on the strip $S := \{z \in \mathbb{C} : 1 \leq \mathrm{Re}(z) \leq 2\}$.

(e) Define $\Omega_0 := \Omega_+$ and $\Omega_{n+1} := \{z \in \mathbb{C} : \mathrm{Re}(z) > -n\} \setminus \{k \in \mathbb{Z} : k \leq 0\}$. Further define

$$\begin{aligned} f_0 : \Omega_0 \to \mathbb{C}, \quad f_0(z) &:= \Gamma_+(z), \\ f_{n+1} : \Omega_{n+1} \to \mathbb{C}, \quad f_{n+1}(z) &:= \frac{f_n(z+1)}{z}. \end{aligned}$$

Show: For all $n \in \mathbb{N}$ the function f_n is complex differentiable and agrees on Ω_k with f_k for all $k \leq n$. Thus there is a complex differentiable function $\Gamma : \Omega := \mathbb{C} \setminus \{k \in \mathbb{Z} : k \leq 0\} \to \mathbb{C}$ with $\Gamma(z) = f_n(z)$ for every $n \in \mathbb{N}$ and all $z \in \Omega_n$.

(f) Show: For each $n \in \mathbb{N}$ one has $\lim_{(z \to -n)} (z+n)\Gamma(z) = \frac{(-1)^n}{n!}$.

For completeness: The Theorem of H. Wieland states the following: Let $\Omega \subseteq \mathbb{C}$ be a domain such that Ω contains the vertical strip S. Then for any function $f : \Omega \to \mathbb{C}$ with

(1) The function f is bounded on S,

(2) The function f satisfies $f(z+1) = z \cdot f(z)$ for all $z \in \Omega$,

one has $f(z) = f(1) \cdot \Gamma(z)$ for all $z \in \Omega$, i. e. the conditions (1) and (2) characterise the Γ-function up to a multiplicative constant.

Hints for solution:

(a) This follows easily because in Ω_+ there is for each compact disk a compakt square containing the disk and for each compact rectangular there is a compact disk containing the rectangular.

(b) We can use the theorem if we check that the requirements are fullfilled. Set $f(z,t) := t^{z-1}e^{-t}$.

(i) Let $z \in \Omega_+$ be a fixed complex number and let $x := \mathrm{Re}(z-1) > -1$. Then we have

$$\begin{aligned} \int_0^\infty |f(z,t)|dt &= \int_0^1 t^x \cdot e^{-t}dt + \int_1^\infty t^x \cdot e^{-t}dt \\ &\leq \int_0^1 t^x \cdot e^{-t}dt + \int_1^\infty C \cdot e^{-\frac{t}{2}}dt \\ &< \infty + \infty = \infty, \end{aligned}$$

where $C > 0$ is a suitable constant satisfiing $t^x e^{-t} \leq C \cdot e^{-\frac{t}{2}}$ on $[1, \infty[$.

(ii) The function $e^{(z-1)\cdot \ln(t)} \cdot e^{-t}$ is for a fixed $t \in]0, \infty[$ complex differentiable.

(iii) Let $K = [a,b] \oplus i \cdot [c,d]$ a compact rectangular in Ω_+. Then we use the same estimation like above: For $(z,t) \in K$ we get

$$\begin{aligned} |f(z,t)| &= |e^{\ln(t)(z-1)} \cdot e^{-t} = |e^{\ln(t)\cdot \mathrm{Re}(z-1)} \cdot e^{-t}| \\ &\leq e^{\max\{\ln(t),0\}\cdot(b-1)} \cdot e^{-t} \\ &= e^{\max\{\ln(t),0\}\cdot(b-1)} \cdot e^{-t} =: g_K(t). \end{aligned}$$

This function is integrable.

Since the requirements of the theorem are fullfilled we get Γ_+ is a complex differentiable function on Ω_+.

(c)

$$\Gamma_+(1) = \int_0^\infty t^0 \cdot e^{-t}dt = 1.$$

$$\Gamma_+(z+1) = \int_0^\infty t^z e^{-t}dt = -t^z e^{-t}|_0^\infty + z \cdot \int_0^\infty t^{z-1} \cdot e^{-t}dt = z\Gamma_+(z).$$

(d) Let $1 \leq \mathrm{Re}(z) \leq 2$. Using analogous decomposition of the Gamma integral we get

$$|\Gamma_+(z)| \leq \int_0^1 |f(z,t)|dt + \int_1^\infty |f(z,t)|dt \leq 1 + \int_1^\infty t \cdot e^{-t}dt =: C < \infty.$$

So the Gamma function is bounded on the strip S.

(e) The definition

$$f_1(z) := \frac{f_0(z)}{z}$$

is on the set Ω_1 a quotient of holomorphic functions, i. e. holomorphic. Thus by iteration we get $\Gamma : \Omega \to \mathbb{C}$ is holomorphic.

It's enough to show $f_{n+1}(z) = f_n(z)$ for all $z \in \Omega_n$. But for $z \in \Omega_n$ we see

$$f_{n+1}(z) = \frac{f_n(z+1)}{z} = \frac{z \cdot f_n(z)}{z} = f_n(z)$$

since the functional equation holds. Thus we get the claim.

(f) We have in Ω:

$$\begin{aligned}
\lim_{z\to -n}(z+n)\Gamma(z) &= \lim_{z\to -n}(z+n)\frac{\Gamma(z+1)}{z} = \ldots \\
&= \lim_{z\to -n}(z+n)\frac{\Gamma(z+n+1)}{z\cdot(z+1)\cdot\ldots\cdot(z+n)} = \lim_{z\to -n}\frac{\Gamma(z+n+1)}{z\cdot(z+1)\cdot\ldots\cdot(z+n-1)} \\
&= \frac{\Gamma(1)}{(-n)\cdot(-n+1)\cdot\ldots\cdot(-1)} = (-1)^n\cdot\frac{1}{n!}.
\end{aligned}$$