Analysis III – Complex Analysis
Hints for solution for the
4. Exercise Sheet

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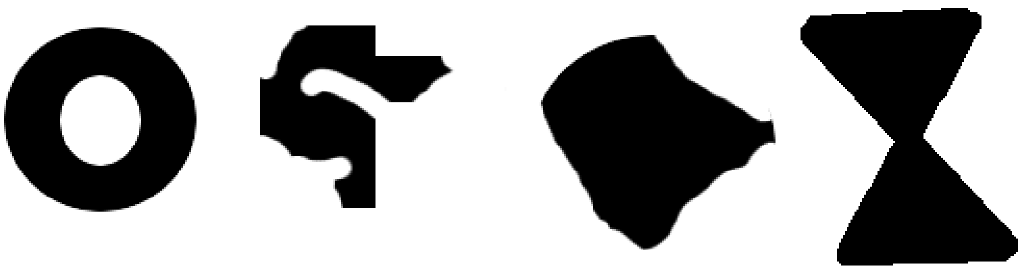
Groupwork

Exercise G1 (Star shaped sets)

(a) Decide which of the scetched subsets of $\mathbb{R}^2$ are star shaped:



a) b) c) d)

(b) Let $X_1 \subseteq \mathbb{R}^n$ and $X_2 \subseteq \mathbb{R}^n$ be two star shaped subsets. Which of the sets $X_1 \cap X_2$, $X_1 \cup X_2$ or $X_1 \times X_2$ are star shaped? Justify your claims.

Remark: You should not use more than 15 minutes for this excercise.

Hints for solution:

(a) Only the sets c) and d) are star shaped (if one looks very carefully perhaps none of the sets are really star shaped...).

(b) It is no problem to find counterexamples in the cases $X \cap Y$ and $X \cup Y$: Use star shaped sets with different central points.
Let X and Y be star shaped. Then there are points $s \in X$ and $t \in Y$ such that for each $x \in X$ there is a linear path $\gamma_x : s \to x$ and for each $y \in Y$ there is a linear path $\gamma_y : t \to y$. Consider the path

$$\gamma : [0,1] \to X \times Y,\ \gamma(a) = (\gamma_x(a), \gamma_y(a)).$$

This path is again a linear path and connects (s,t) with (x,y). Since (x,y) is arbitrary in $X \times Y$, the set $X \times Y$ is star shaped with center (s,t).

Exercise G2 (Simple connected sets)

(a) Let (X,d_1) and (Y,d_2) be metric spaces and $\varphi : X \to Y$ a homeomorphism. Show: If $A \subseteq X$ is simply connected, then $\varphi(A) \subseteq Y$ is simply connected.

(b) Let $0 < r < R$ be real numbers. Scetch the set

$$K_{r,R} := \{x \in \mathbb{R}^2 : \ r < \|x\| < R\} \setminus \{(0,y) \in \mathbb{R}^2 : \ y \leq 0\}$$

for a suitable choice of r and R. Use polar coordinates to prove that $K_{r,R}$ is simply connected.

(c) Show: The sets $\mathbb{R}^2 \setminus \{0\}$ and $\mathbb{S}^1 := \{x \in \mathbb{R}^2 : \ \|x\| = 1\}$ are not simply connected.

Hint: For (c) consider a useful vector field and use the homotopy invariance of the path integral.

Hints for solution:

(a) Let $A \subseteq X$ be simply connected and define $B := \varphi(A)$. Let γ_1 and γ_2 be loops in B. Since φ is a homeomorphism $\varphi^{-1}(\gamma_1)$ and $\varphi^{-1}(\gamma_2)$ are loops in A. Since A is simply connected these paths are homotopic by a homotopy H. Then $\varphi(H)$ is a homotopy in B that means γ_1 and γ_2 are homotopic.

(b) The set $K_{r,R}$ is exactly the image of the polar coordinate transformation of the set

$$\Omega :=]r,R[\times]-\pi,\pi[.$$

On this domain the polar transformation is a homeomorphism onto its image. Since Ω is convex, it is simple connected. This fact implies the simple connectedness of $K_{r,R}$ by (a).

(c) Use the winding vector field of the third exercise sheet:

$$f(x,y) = \frac{1}{x^2+y^2} \cdot \begin{pmatrix} -y \\ x \end{pmatrix}.$$

Since there are loop integrals which don't vanish these sets can't be simply connected.

Notes on Homotopy

Let (X,d) be a pathwise connected metric space. For $x \in X$ we use the notation γ_x for the path $\gamma_x : [0,1] \to X$, $\gamma_x(t) = x$ for all $t \in [0,1]$. If γ_1 and γ_2 are paths in X which are homotopic we write $H : \gamma_1 \to \gamma_2$ if H is a homotopy $H : [0,1] \times [0,1] \to X$ for γ_1 and γ_2, i. e. $H(0,t) = \gamma_1(t)$ and $H(1,t) = \gamma_2(t)$ for all $t \in [0,1]$.
Further you can use the theorems of the lectures although a homotopy H is usually not continuously differentiable in this excercise sheet: Treat continuous homotopies like $\mathscr{C}^2$-homotopies. Especially you can use the homotopy invariance of the path integral.

Exercise G3 (Homotopy)

(a) Let (X,d) be a pathwise connected metric space and let $x, y \in X$ be arbitrary points. Show that x-y-homotopy defines an equivalence relation on the set $\Gamma(X,x,y)$ of all paths starting in x and ending in y.

(b) Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Show that the following conditions are equivalent:
   (i) The set Ω is connected.
   (ii) For every $x, y \in \Omega$ the path γ_x is homotopic to γ_y.

Hints for solution:

(a) Reflexivity and symmetry are not difficuilt to prove. For proving transitivity one could build

$$H(s,t) := \begin{cases} H_1(2s,t) : & 0 \le s \le \frac{1}{2} \\ H_2(2s-1,t) : & \frac{1}{2} \le s \le 1 \end{cases}$$

where $H_1 : \gamma_0 \to \gamma_1$ and $H_1 : \gamma_1 \to \gamma_2$. This new map H is continuous hence a homotopy $H : \gamma_0 \to \gamma_2$

(b) Assume Ω is connected. Since Ω is open it is pathwise connected. Let $x, y \in \Omega$ be arbitrary. Then there is a path $\gamma : x \to y$. Define the homotopy

$$H(s,t) := \gamma(s).$$

This is a homotopy $\gamma_x \simeq \gamma_y$.
Assume for arbitrary $x, y \in \Omega$ there is a homotpy $H : \gamma_x \simeq \gamma_y$. Define

$$\gamma(t) := H\left(t, \frac{1}{42}\right).$$

Then γ is a path from x to y. This means Ω is pathwise connected hence connected.

Exercise G4 (Homotopy classes of loops on the circle)
We consider the set

$$\pi_1(\mathbb{T},1) := \{[\gamma]_{\simeq} : \ \gamma : [0,1] \to \mathbb{T} \text{ is a continuous path with } \gamma(0) = \gamma(1) = 1\}$$

where $\gamma_1 \simeq \gamma_2$ if and only if there is a homotopy $H : \gamma_1 \to \gamma_2$ with $H(s,0) = H(s,1) = 1$ for all $s \in [0,1]$. Prove that the set $\pi_1(\mathbb{T},1)$ has infinitely many elements.
We know, e. g. from Analysis II, that the set $\pi_1(\mathbb{T},1)$ forms a group with the multiplication $[\gamma_1] \cdot [\gamma_2] := [\gamma_1 + \gamma_2]$ and inversion $[\gamma^{-1}] := [-\gamma]$. Find an isomorphic copy of $\mathbb{Z}$ in $\pi_1(\mathbb{T},1)$.
Remark: In fact the group $\pi_1(\mathbb{T},1)$ is isomorphic to $\mathbb{Z}$ but we can't prove this without further analysis, e. g. the analysis of covering spaces.

Hints for solution: Integrate the function $f(z) := \frac{1}{z}$ by the path $\beta_n(t) := e^{2\pi i n t}$. This leads to

$$\int_{\beta_n} f \, dt = 2\pi n \cdot i.$$

Since f satisfies the integrability condition it is locally exact and thus the integral is constant on a homotopy class. It means there are at least $|\mathbb{Z}|$ different homotopy classes of paths in $\mathbb{T}$.
Use $[\beta_m] \cdot [\beta_n] = [\beta_{m+n}]$ and the path integral argument: The element $[\beta_1] \in \pi_1(\mathbb{T})$ generates a subgroup of $\pi_1(\mathbb{T})$ which is isomorphic to $\mathbb{Z}$.
Of course the element $[\beta_{42}]$ generates a subgroup isomorphic to $\mathbb{Z}$, too. This corresponds to the subgroup $42\mathbb{Z}$ of $\mathbb{Z}$. Nice to know but not easy to prove: $\mathbb{Z} \ni n \to [\beta_n] \in \pi_1(\mathbb{T})$ is an isomorphism of groups.

Homework

Exercise H1 (Complex path integrals) (1 point)

(a) Determine the following complex path integrals where every path is counterclockwise orientated.

   (i) $\int_{|z|=1} e^{-z^2} dz$.

   (ii) $\int_\gamma \overline{z} dz$, where γ describes the triangle with the endpoints $i, 1-i$ and $-1-i$.

(b) From the lectures we know that every complex path integral can be decomposed into two real path integrals

$$\int_\gamma f\,dz = \int_\gamma \omega_1 ds + i \cdot \int_\gamma \omega_2 ds.$$

Determine for $f(z) = \frac{1}{z}$ the vector fields ω_1 and ω_2. Check the integrability conditions for ω_1 and ω_2 and discuss the existence of primitives of these vector fields.

Hints for solution:

(i) The result is 0.

(ii) The result is $4i$.

(b)

$$\omega_1(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix} x \\ y \end{pmatrix} \quad \omega_2(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}.$$

The vector field ω_1 has a primitive on the whole $\mathbb{R}^2 \setminus \{0\}$. The vector field ω_2 is well known to have locally primitives but no global primitive on the whole $\mathbb{R}^2 \setminus \{0\}$. Both vector fields satisfy the integrability conditions.

Exercise H2 (Fundamental Theorem of Algebra) (1 point)

For a real number $r \in [0, \infty[$ we consider the path $\alpha_r : [0,1] \to \mathbb{C}$, $\alpha_r(t) := r \cdot e^{2\pi i t}$ and for a natural number $n \in \mathbb{N}$ we consider the path $\beta_n : [0,1] \to \mathbb{T}$, $\beta_n(t) := e^{2\pi i n t}$. Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial of the form

$$p(z) = z^n + \sum_{k=0}^{n-1} a_k \cdot z^k$$

which has no roots, i. e. there is no point $z_0 \in \mathbb{C}$ with $p(z_0) = 0$.

(a) Let $r > 0$ be a fixed positive real number. Show: The loop $\gamma_r(t) := p(\alpha_r(t))$ is homotopic in $\mathbb{C} \setminus \{0\}$ to the loop $\gamma_0(t) := p(\alpha_0(t))$.

(b) Show: There is a real number $r \geq 1$ such that none of the polynomials

$$f_q(z) := z^n + q \cdot \sum_{k=0}^{n-1} a_k z^k, \quad 0 \leq q \leq 1$$

has a root on $r \cdot \mathbb{T} := \{z \in \mathbb{C} : |z| = r\}$.

(c) Use (b) to find a real number $r \geq 1$ and a homotopy $H : \gamma_r \to r^n \cdot \beta_n$ in $\mathbb{C} \setminus \{0\}$ where n is the degree of p.

(d) Show: p has to be a constant polynomial.

(e) Show the Fundamental Theorem of Algebra: Every complex polynomial f which has no root in $\mathbb{C}$ is constant.

The first complete proof of the Fundamental Theorem of Algebra was given by C. F. Gauß in 1799. During his life Gauß gave three further proofs of this theorem and the following idea of a proof is accredited to Gauß, too:
Let p a complex polynomial of degree n. This polynomial maps loops into loops. If one looks on the image of a loop α_r of very small radius r then the image loop lies in a small neighbourhood of a_0, the absolute part of p. If one looks on a loop of very large radius the image loop behaves like $a_n \cdot r^n \cdot \beta_n$ where n is the degree of p and a_n the highest coefficient of p. Especially it winds n times around the origin. If one varies the radius r continuously the image loops varies continuously in the complex plane: The image loops have to hit the origin, elsewhere the origin can't pass from the exterior of the loops of small radius into the interior of the loops of large radius. So p must have a root.

(f*) Explain shortly in which way our proof in this excercise makes this idea precise.

Hints for solution:

(a) Since the polynomial has no root the map

$$H : [0, r] \times [0,1] \ni (s,t) \to p(\alpha_s(t)) \in \mathbb{C} \setminus \{0\}$$

is the desired homotopy of loops.

(b) Choose $r > \max\{1, |a_0| + |a_1| + \ldots + |a_{n-1}|\}$. Then we see for $|z| = r$:

$$\begin{aligned} |f_q(z)| &\geq |z^n| - \sum_{k=0}^{n-1} |a_k||z^k| \\ &\geq |z^n| - |z^{n-1}| \cdot \sum_{k=0}^{n-1} |a_k| \\ &> |z^n| - |z^{n-1}| \cdot |z| = r^n - r^n = 0. \end{aligned}$$

(c) Choose r like in (b). We see the map

$$H(s,t) := f_s(\alpha_r(t)) \in \mathbb{C} \setminus \{0\}$$

is a homotopy $H : f_0 \circ \alpha_r \to f_1 \circ \alpha_r$. Since $f_0 \circ \alpha_r = r^n \cdot \beta_n$ and $f_1 \circ \alpha_r = \gamma_r$ we are done.

(d) From (a) and (c) we conclude that the constant path γ_0 is homotopic to the path $r^n \cdot \beta_n$ in $\mathbb{C} \setminus \{0\}$. This means $n = 0$ and we conlcude p is constant. Elsewhere the constant path β_0 would be homotopic to the path β_n, because $r^n \cdot \beta_n \simeq \beta_n$ and $\beta_0 \simeq \gamma_0$, a contradiction.

(e) Assume f is a polynomial with degree n and no root. We get

$$f(z) = a_n \cdot z^n + \sum_{k=0}^{n-1} a_k z^k.$$

Since $a_n \neq 0$ we can form

$$p(z) := \frac{f(z)}{a_n}.$$

This is again a polynomial without root and we conclude from (d) p is constant. This means f is constant.

Exercise H3 (Browers fixed point theorem in dimension 2) (1 point)

Remember the unit disk is defined by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

(a) Prove: There is no continuous map $f : \overline{\mathbb{D}} \to \mathbb{T}$ with $f(z) = z$ for every $z \in \mathbb{T}$.
Hint: If one has a contiuous map $f : X \to Y$ the composition of f with a homotopy in X is a homotopy in Y.

(b) Prove Browers fixed point theorem in dimension 2: Every continuous map $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ has at least one fixed point, i. e. there is a point $z_0 \in \overline{\mathbb{D}}$ with $f(z_0) = z_0$.
Hint: For $z \in \overline{\mathbb{D}}$ consider the set $\{f(z) + \lambda \cdot (z - f(z)) : \lambda > 0\} \cap \mathbb{T}$. This set has exactly one element if f has no fixed point. We call the element of this set $h(z)$. You can use without a proof that $h : \overline{\mathbb{D}} \to \mathbb{T}$ is well defined and continuous.
In which way $h(z)$ depends on z and $f(z)$? Scetch it for some example.

Hints for solution:

(a) Assume there is such a map f. We build the homotopy

$$H(s,t) := f(s \cdot e^{2\pi i t}).$$

Since this is a homotopy in $\mathbb{T}$ from some the constant path to the path β_1 we get a contradiction: Such a map f cannot exist.

(b) Assume f has no fixed point. For $|z| = 1$ we get

$$\{\lambda \cdot z + (1 - \lambda) \cdot f(z) : \lambda > 0\} \cap \mathbb{T} = \{z\}.$$

This means h is a continuous map which fixes $\mathbb{T}$ pointwise. This is a contradiction to (a). We follow that f has a fixed point.
It is of course possible to write down the map h explicitely...