# Analysis III – Complex Analysis
# Hints for solution for the
# 3. Exercise Sheet


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## Groupwork

**Exercise G1** (The standard estimation)

Let $f: \mathbb{R}^n \supseteq \Omega \to \mathbb{R}^n$ be a continuous vector field and $\gamma : [0,1] \to \Omega$ a piecewise continuously differentiable path. Show the following estimation:

$$\left| \int_\gamma f \, ds \right| \leq \max \left\{ \left\| f(\gamma(t)) \right\|_2 : 0 \leq t \leq 1 \right\} \cdot L(\gamma),$$

where $L(\gamma)$ denotes the legth of $\gamma$.

**Hints for solution:** Since $\gamma : [0,1] \to \Omega$ is continuous the function $[0,1] \ni t \to \left\| f(\gamma(t)) \right\| \in \mathbb{R}$ is continuous and has a maximum $K \in \mathbb{R}$, cause $[0,1]$ is compact. So we see

$$\begin{aligned} \left| \int_\gamma f \, ds \right| &= \left| \int_0^1 \langle f(\gamma(t), \gamma'(t) \rangle dt \right| \\ &\leq \int_0^1 \left| \langle f(\gamma(t), \gamma'(t) \rangle \right| dt \\ &\leq \int_0^1 \left\| f(\gamma(t) \right\|_2 \cdot \left\| \gamma'(t) \right\|_2 dt \\ &\leq \int_0^1 K \cdot \left\| \gamma'(t) \right\|_2 dt \\ &= K \cdot L(\gamma). \end{aligned}$$

**Exercise G2** (Winding around the origin)

Consider the vector field

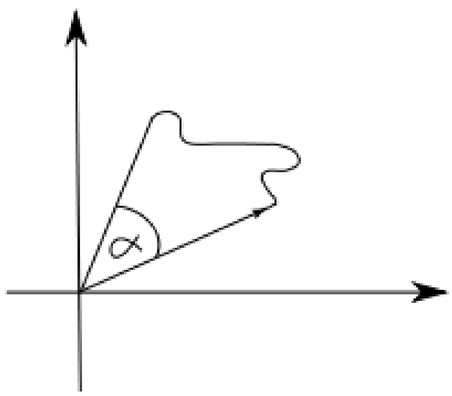
$$f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2, \quad f\begin{pmatrix} x \\ y \end{pmatrix} := \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$$

and the two star shaped domains $G_1 := \mathbb{R}^2 \setminus \{(x,0) : x \leq 0\}$ and $G_2 := \mathbb{R}^2 \setminus \{(x,0) : x \geq 0\}$.

(a) Show that $f$ has a potential on $G_1$ and a potential on $G_2$ and determine them.
**Hint:** Use the polar decomposition: $\gamma(t) = r(t) \cdot \begin{pmatrix} \cos(\varphi(t)) \\ \sin(\varphi(t)) \end{pmatrix}$.

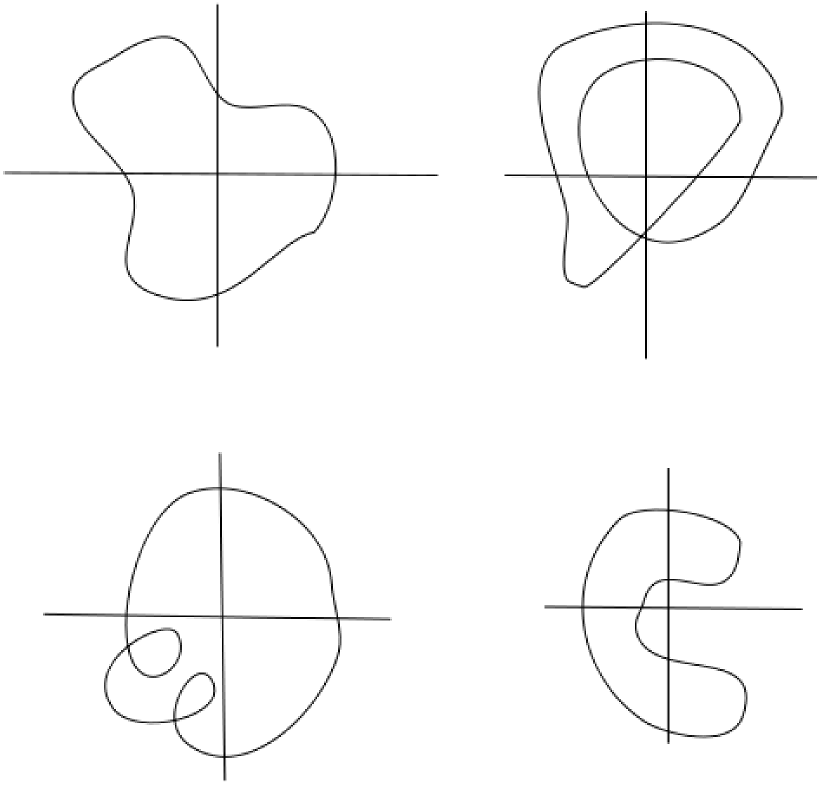
(b) Is there a global potential for $f$ on $\mathbb{R}^2 \setminus \{0\}$?

(c) Consider paths of the following form:




Show for the (improper) path integral that $\int_\gamma f\, ds = \alpha$, where $\alpha \in [0, 2\pi[$ is the included angle of the path.

(d) Determine the path integral for the following counterclockwise parametrised curves:



The number $\frac{1}{2\pi}\int_\gamma f\,ds$ is called the *winding number* of $\gamma$ in 0 for a loop $\gamma$ . Why?

**Hints for solution:**

(a) The polar decomposition $\Phi$ induces diffeomorphisms $\Phi_1 : G_1 \to ]0,\infty[\times]-\pi,\pi[$ and $\Phi_2 : G_2 \to ]0,\infty[\times]0,2\pi[$.
Using this fact any loop $\gamma$ in $G_1$ or $G_2$ resp. has a unique representation

$$\gamma(t) = r(t)\cdot\begin{pmatrix}\cos(\varphi(t))\\ \sin(\varphi(t))\end{pmatrix}$$

where $r$ is a loop in $]0,\infty[$ and $\varphi$ is a loop in $]-\pi,\pi[$ or $]0,2\pi[$ resp. We get

$$\begin{aligned}\int_\gamma f\,dt &= \int_0^1 \langle f(\gamma(t)),\gamma'(t)\rangle dt\\ &= \int_0^1 \left\langle \frac{r(t)}{r(t)^2}\begin{pmatrix}-\sin(\varphi(t))\\ \cos(\varphi(t))\end{pmatrix}, r(t)\cdot\varphi'(t)\cdot\begin{pmatrix}-\sin(\varphi(t))\\ \cos(\varphi(t))\end{pmatrix}+r'(t)\cdot\begin{pmatrix}\cos(\varphi(t))\\ \sin(\varphi(t))\end{pmatrix}\right\rangle dt\\ &= \int_0^1 \varphi'(t)dt = \varphi(1)-\varphi(0) = 0.\end{aligned}$$

So every loop integral vanishes. That implies the existence of primitives $F_1$ and $F_2$. An analogous calculation shows

$$F_1(x,y) = \varphi,$$

$$F_2(x,y) = \varphi - \pi$$

where $\varphi$ is the argument of $(x,y)$ in its polar decomposition.
For finding the primitives: We integrate a path starting in $(1,0)$ and ending in $(x,y)$ for $F_1$ and a path starting in $(-1,0)$ and ending in $(x,y)$ for $F_2$.

(b) Consider the path $\gamma(t) = \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix}$. We can decompose this loop in a sum of four paths $\gamma_1, \gamma_2, \gamma_3$ and $\gamma_4$, describing a quarter of the path $\gamma$: $\gamma_1 : (1,0) \to (0,1)$, $\gamma_2 : (0,1) \to (-1,0)$, $\gamma_3 : (-1,0) \to (0,-1)$ and $\gamma_4 : (0,-1) \to (1,0)$. This makes it possible to calculate the path integral by using both primitives $F_1$ and $F_2$, since every sub path is completely in $G_1$ or in $G_2$. Using $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ We see

$$\int_\gamma f\, ds = 2\pi.$$

There can't exist a potential for $f$ on $\mathbb{R}^2 \setminus \{0\}$.

(c) In (a) we see the path integral $\int_\gamma f\, ds$ vanishes for paths with constant $\varphi(t)$. So the value of the improper path integral is $\alpha$.

(d) $2\pi$, $4\pi$, $2\pi$, $0$.

**Exercise G3** (Path integrals and potentials)
In the lectures we will see that a vector field $f : \mathbb{R}^n \to \mathbb{R}$ has a potential if and only if its Jacobian $J_f(x)$ is symmetric for all $x \in \mathbb{R}^n$.
Decide whether the following vector fields have a potential. Determine the potential if it exists.

$$\begin{array}{rrcl} f : \mathbb{R}^2 \to \mathbb{R}^2, & f(x,y) & := & (2xy^3 + \cos(x),\ 3x^2y^2 + \cos(y))^T, \\ g : \mathbb{R}^3 \to \mathbb{R}^3, & g(x,y,z) & := & (1 + y(1+x),\ x(1+z),\ xy)^T, \\ h : \mathbb{R}^2 \to \mathbb{R}^2, & h(x,y) & := & (y \cdot e^{xy},\ x \cdot e^{xy} + 1)^T. \end{array}$$

**Hints for solution:** Only $f$ and $h$ have a potential:

$$\begin{aligned} F(x,y) &= x^2 \cdot y^3 + \sin(x) + \sin(y) \\ H(x,y) &= e^{x\cdot y} + y. \end{aligned}$$

Of course the potentials are only unique up to a additive constant.
Compute $\partial_y g_1 = 1 + x$ and $\partial_x g_2 = 1 + z$. We see, the Jacobian $J_g(0,0,1)$ is not symmetric.

## Homework

**Exercise H1** (Equivalence of paths) (1 point)

(a) Show that the equivalence of paths is an equivalence relation.

(b) Let $f : \mathbb{R}^n \supseteq \Omega \to \mathbb{R}^n$ be a continuous vector field. Prove that for equivalent paths $\gamma_1 : [a,b] \to \Omega$ and $\gamma_2 : [c,d] \to \Omega$ one has $\int_{\gamma_1} f\,ds = \int_{\gamma_2} f\,ds$.

**Hints for solution:**

(a) Easy

(b) Let $\gamma : [a,b] \to \Omega$ and $\varphi : [a,b] \to [c,d]$ be a diffeomorphism with positive derivative. Then we see

$$\begin{aligned}
\int_{\gamma_1} f\,ds &= \int_{\gamma_2\circ\varphi} f\,ds \\
&= \int_a^b \left\langle f(\gamma_2(\varphi(t))), \gamma_2'(\varphi(t)) \right\rangle \cdot \varphi'(t)dt \\
&= \int_{\varphi(a)}^{\varphi(b)} \left\langle f(\gamma_2(t)), \gamma_2'(t) \right\rangle dt \\
&= \int_{\gamma_2} f\,ds.
\end{aligned}$$

**Exercise H2** (Connectedness and pathwise connectedness) (1 point)

(a) Let $\Omega \subseteq \mathbb{R}^n$ be open and fix a point $x \in \Omega$. Consider the set $G_x := \{y \in \Omega :$ there is a continuous curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y\}$. Show that $G_x$ is open and closed in $\Omega$, i. e. open and closed in the metric space $(\Omega, \|\cdot\|_2)$.

(b) Let $\Omega \subseteq \mathbb{R}^n$ be open. Prove that $\Omega$ is connected if and only if $\Omega$ is pathwise connected.

(c) Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Show that for every $x, y \in \Omega$ there is a piecewise linear path $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$.

**Hints for solution:**

(a) We show $G_x$ is open. Let $y \in G_x$ be arbitrary. Then there is an $\varepsilon > 0$ such that $K_\varepsilon(y) \subseteq \Omega$. Since $K_\varepsilon(y)$ is convex, $K_\varepsilon(y)$ is pathwise connected: For every $z \in K_\varepsilon(y)$ there is a path $\gamma_z : [0,1] \to \Omega$ which starts in $y$ and ends in $z$. Since there is a path $\gamma_y$ starting in $x$ and ending in $y$, the path $\gamma := \gamma_z + \gamma_y$ is a path starting in $x$ and ending in $z$.

We show $G_x$ is closed in $\Omega$. Let $y \in \Omega$ with $y \in \overline{G_x}$. Since $\Omega$ is open, there is a neighbourhood $K_\varepsilon(y) \subseteq \Omega$. Of course $K_\varepsilon(y) \cap G_x$ is not empty, cause $y$ is a accumulation point of $G_x$. Since $K_\varepsilon(y)$ is convex, we conclude the assertion.

(b) We have only to show that connectedness implies pathwise connectedness. By (a) we see that for every $x \in \Omega$ the set $G_x$ is open and closed in $\Omega$. Since $\Omega$ is connected we follow $G_x = \emptyset$ or $G_x = \Omega$. With $x \in G_x$ the assertion follows.

(c) Let $x, y \in \Omega$ be arbitrary points. Since $\Omega$ is pathwise connected there is a path $\gamma : [0,1] \to \Omega$ starting in $x$ and ending in $y$. Let $K := \gamma([0,1])$. Of course $K$ is compact. We define an open covering: Choose for each $t \in [0,1]$ a convex open set $U_t \in \Omega$ with $\gamma(t) \in U_t$. This is possible because $\Omega$ is open. The family $(U_t)_{t\in[0,1]}$ covers $K$. From compactness we conclude: There are finitely many $t_1 < t_2 < \ldots < t_{n-1}$ points in $[0,1]$ with $K \subseteq U_{t_1} \cup \ldots \cup U_{t_{n-1}}$. Choose $t_0 = 0$ and $t_n = 1$ we of course have

$$K \subseteq U_0 \cup U_{t_1} \cup \ldots \cup U_{t_{n-1}} \cup U_1 \subseteq \Omega.$$

Since every $U_{t_k}$ is convex and has non empty intersection with $U_{t_{k+1}}$ we can choose points

$$x_k \in U_k \cap U_{k+1}.$$

The piecewise linear path $\gamma_{new} : x \to x_0 \to x_1 \to x_2 \to \ldots \to x_{n-1} \to y$ is contained in $U_0 \cup U_1 \cup \bigcup_{k=1}^{n-1} U_k$. So we are done.

**Exercise H3** (Potentials) (1 point)

(a) Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $f : \Omega \to \mathbb{R}^n$ be a continuous vector field. Assume $F_1 : \Omega \to \mathbb{R}$ and $F_2 : \Omega \to \mathbb{R}$ are potentials for $f$. Show that $F_1 - F_2$ is a constant function.

(b) Let $\Omega$ be an arbitrary nonempty open subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$ be a continuous vector field. Assume $F_1 : \Omega \to \mathbb{R}$ and $F_2 : \Omega \to \mathbb{R}$ are potentials for $f$. What can you say about the difference function $F_1 - F_2$?

**Hints for solution:**

(a) Assume $\nabla F_1 = \nabla F_2$ and let $G := F_1 - F_2$. We have $\nabla G = 0$ and have to show that $G$ is constant. Fix a point $x \in \Omega$ and consider the set

$$U := \{y \in \Omega : \; G(y) = G(x)\}.$$

Of course $U$ is a closed set in $\Omega$. We show that $U$ is open: Let $y \in U$ arbitrary. Since $\Omega$ is open there is a $\varepsilon$-neighbourhood $K_y := K_\varepsilon(y)$ with $K_y \subseteq \Omega$. Since $K_y$ is convex we can apply the Schrankensatz: For each $z \in K_y$ we have

$$\left\|G(y) - G(z)\right\| \leq \left\|y - z\right\| \cdot \max_{\xi \in \{ty+(1-t)z:\; 0 \leq t \leq 1\}} \|\nabla G(\xi)\| = 0.$$

We see $K_y \subseteq U$ so $U$ is indeed open. Since $U$ is not empty we conclude $U = \Omega$ cause $\Omega$ is connected. This means $G$ is constant.

(b) The function is locally constant by (a). Locally constant means, on every pathwise connected open subset $U$ of $\Omega$ we have $G_{|_U}$ is a constant function. For example the function $f : ]0,1[ \cup ]42,\infty[ \to \mathbb{R}$,

$$f(x) = \begin{cases} 42 & x \in ]0,1[, \\ 0 & x \in ]42,\infty[ \end{cases}$$

is locally constant.