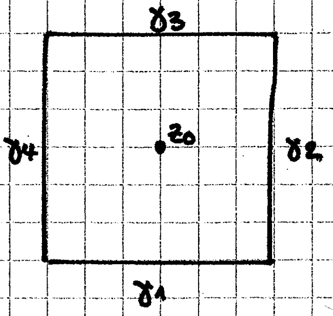
(G2) •



$$\gamma_1 : t \mapsto \begin{pmatrix} x_0 - \varepsilon/2 \\ y_0 - \varepsilon/2 \end{pmatrix} + t \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix} \quad t \in [0,1]$$
$$\gamma_2 : t \mapsto \begin{pmatrix} x_0 + \varepsilon/2 \\ y_0 - \varepsilon/2 \end{pmatrix} + t \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} \quad t \in [0,1]$$
$$\gamma_3 : t \mapsto \begin{pmatrix} x_0 + \varepsilon/2 \\ y_0 + \varepsilon/2 \end{pmatrix} + t \begin{pmatrix} -\varepsilon \\ 0 \end{pmatrix} \quad t \in [0,1]$$
$$\gamma_4 : t \mapsto \begin{pmatrix} x_0 - \varepsilon/2 \\ y_0 + \varepsilon/2 \end{pmatrix} + t \begin{pmatrix} 0 \\ -\varepsilon \end{pmatrix} \quad t \in [0,1]$$

$$\int_\gamma f\,ds = \sum_{i=1}^4 \int_{\gamma_i} f\,ds$$
$$= \int_0^1 f(\gamma_1(t)) \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix} dt + \int_0^1 f(\gamma_2(t)) \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} dt + \int_0^1 f(\gamma_3(t)) \begin{pmatrix} -\varepsilon \\ 0 \end{pmatrix} dt + \int_0^1 f(\gamma_4(t)) \begin{pmatrix} 0 \\ -\varepsilon \end{pmatrix} dt$$
$$= \int_0^1 f_1\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \varepsilon \begin{pmatrix} t-1/2 \\ -1/2 \end{pmatrix}\right)\varepsilon - f_1\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \varepsilon \begin{pmatrix} t-1/2 \\ -1/2 \end{pmatrix}\right)\varepsilon$$
$$+ f_2\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \varepsilon \begin{pmatrix} 1/2 \\ t-1/2 \end{pmatrix}\right)\varepsilon - f_2\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \varepsilon \begin{pmatrix} 1/2 \\ t-1/2 \end{pmatrix}\right)\varepsilon \quad dt$$

$$\frac{1}{\varepsilon^2} \int_\gamma f\,ds = \frac{1}{\varepsilon} \int_0^1 f_1\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \varepsilon \begin{pmatrix} t-1/2 \\ -1/2 \end{pmatrix}\right) - f_1\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \varepsilon \begin{pmatrix} t-1/2 \\ -1/2 \end{pmatrix}\right)$$
$$+ f_2\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \varepsilon \begin{pmatrix} 1/2 \\ t-1/2 \end{pmatrix}\right) - f_2\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \varepsilon \begin{pmatrix} 1/2 \\ t-1/2 \end{pmatrix}\right) dt$$

Nullen einschieben

$$= \int_0^1 \frac{f_1\left(z_0 + \varepsilon \begin{pmatrix} t-1/2 \\ -1/2 \end{pmatrix}\right) - f_1(z_0)}{\varepsilon - 0} + \frac{f_1(z_0) - f_1\left(z_0 - \varepsilon \begin{pmatrix} t-1/2 \\ -1/2 \end{pmatrix}\right)}{0 - (-\varepsilon)}$$
$$+ \frac{f_2\left(z_0 + \varepsilon \begin{pmatrix} 1/2 \\ t-1/2 \end{pmatrix}\right) - f_2(z_0)}{\varepsilon - 0} + \frac{f_2(z_0) - f_2\left(z_0 - \varepsilon \begin{pmatrix} 1/2 \\ t-1/2 \end{pmatrix}\right)}{0 - (-\varepsilon)} \quad dt$$

Richt'abl. erkennen

$$\lim_{\varepsilon \to 0} \int_\gamma f\,ds = \int_0^1 \left(\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}\right)(z_0) \cdot \begin{pmatrix} t-1/2 \\ -1/2 \end{pmatrix} + \left(\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}\right)(z_0) \cdot \begin{pmatrix} t-1/2 \\ -1/2 \end{pmatrix}$$
$$+ \left(\frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial y}\right)(z_0) \begin{pmatrix} 1/2 \\ t-1/2 \end{pmatrix} + \left(\frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial y}\right)(z_0) \begin{pmatrix} 1/2 \\ t-1/2 \end{pmatrix} \quad dt$$
$$= 2 \int_0^1 \frac{\partial f_1}{\partial x}(z_0)\left(t - \frac{1}{2}\right) - \frac{\partial f_1}{\partial y}(z_0) \cdot \frac{1}{2} + \frac{\partial f_2}{\partial x}(z_0) \cdot \frac{1}{2} + \frac{\partial f_2}{\partial y}(z_0)\left(t - \frac{1}{2}\right) dt$$
$$= \cancel{2} \cdot \left[\frac{\partial f_1}{\partial x}(z_0)\left(\frac{t^2}{\cancel{2}} - \frac{t}{\cancel{2}}\right) - \frac{\partial f_1}{\partial y}(z_0) \frac{t}{\cancel{2}} + \frac{\partial f_2}{\partial x}(z_0) \frac{t}{\cancel{2}} + \frac{\partial f_2}{\partial y}(z_0)\left(\frac{t^2}{\cancel{2}} - \frac{t}{\cancel{2}}\right) \right]_0^1$$
$$= -\frac{\partial f_1}{\partial y}(z_0) + \frac{\partial f_2}{\partial x}(z_0)$$

- Warum bezeichnet man $\operatorname{rot} f(z_0)$ als Rotation von f in z_0?

  γ ist ein geschl. Weg mit Zentrum z_0; dh wir summieren die VF-Komponenten in Richtung γ längs γ auf, dividieren anschl. durch die Fläche und lassen dann den Weg immer enger um z_0 laufen – wir berechnen also eine Rotationsdichte (infinitesimal) im Punkt z_0.

(G3) Zunächst stellen wir fest, daß R kompakt ist, f und seine partiellen Ableitungen stetig sind, wir können also den Satz von Fubini anwenden (*)

$$\int_c^d \int_a^b \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \, dx\, dy \overset{(*)}{=} \int_c^d \int_a^b \frac{\partial f_2}{\partial x} \, dx\, dy - \int_a^b \int_c^d \frac{\partial f_1}{\partial y} \, dy\, dx$$

$$= \int_c^d f_2(b,y) - f_2(a,y) \, dy - \int_a^b f_1(x,d) - f_1(x,c) \, dx = (\Delta)$$

Wir setzen

$$\left.\begin{aligned} \gamma_1(t) &= \binom{b}{0} + t\binom{0}{1}, \ t \in [c,d] \\ \gamma_2(t) &= \binom{a}{0} + t\binom{0}{1}, \ t \in [c,d] \end{aligned}\right\} \gamma_i'(t) = \binom{0}{1}$$

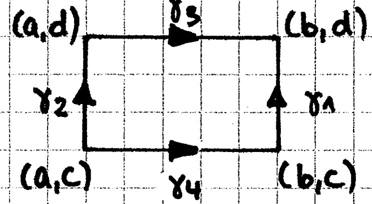
$$\left.\begin{aligned} \gamma_3(t) &= \binom{0}{d} + t\binom{1}{0}, \ t \in [a,b] \\ \gamma_4(t) &= \binom{0}{c} + t\binom{1}{0}, \ t \in [a,b] \end{aligned}\right\} \gamma_i'(t) = \binom{1}{0}$$

und erhalten somit

$$(\Delta) = \int_c^d f(\gamma_1(t))\gamma_1'(t)\, dt - \int_c^d f(\gamma_2(t))\gamma_2'(t)\, dt$$

$$- \int_a^b f(\gamma_3(t))\gamma_3'(t)\, dt + \int_a^b f(\gamma_4(t))\gamma_4'(t)\, dt$$

$$= \int_{\gamma_1} f\, ds - \int_{\gamma_2} f\, ds - \int_{\gamma_3} f\, ds + \int_{\gamma_4} f\, ds$$



$$= \int_\gamma f\, ds$$