Analysis III – Complex Analysis
Hints for solution for the
2. Exercise Sheet

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Groupwork

Exercise G1 (Cauchy-Riemann differential equations I)

Consider the function $f(z) := e^z$. Use the Cauchy-Riemann differential equations to prove that f is differentiable on the whole complex plane.

Hints for solution: We compute the real vector field for f:

$$\begin{aligned} F(x,y) &= \begin{pmatrix} \mathrm{Re} f(x+y\cdot i) \\ \mathrm{Im} f(x+y\cdot i) \end{pmatrix} = \begin{pmatrix} \mathrm{Re}(e^x\cdot e^{yi}) \\ \mathrm{Im}(e^x\cdot e^{yi}) \end{pmatrix} \\ &= \begin{pmatrix} e^x\cdot cos(y) \\ e^x\cdot \sin(y) \end{pmatrix}. \end{aligned}$$

The Jacobian von F is of course:

$$J_F(x,y) = e^x \cdot \begin{pmatrix} \cos(y) & -\sin(y) \\ \sin(y) & \cos(y) \end{pmatrix}.$$

The Cauchy-Riemann differential equations are obviously satisfied.

Exercise G2 (Cauchy-Riemann differential equations II)

Consider the function $f(x+y\cdot i) := x^3\cdot y^2 + x^2\cdot y^3\cdot i$ defined on the whole complex plane. Determine the subset $\Omega\subseteq\mathbb{C}$ on which f has a complex derivative. Is there an inner point $z_0\in\Omega$?

Hints for solution: We compute the real vector field for f:

$$F(x,y) = \begin{pmatrix}\mathrm{Re} f(x+y\cdot i)\\ \mathrm{Im} f(x+y\cdot i)\end{pmatrix} = \begin{pmatrix}x^3\cdot y^2\\ x^2\cdot y^3\end{pmatrix}$$

The Jacobian von F is of course:

$$J_F(x,y) = \begin{pmatrix}3x^2\cdot y^2 & 2x^3\cdot y\\ 2x\cdot y^3 & 3x^2\cdot y^2\end{pmatrix}.$$

The Cauchy-Riemann differential equations are satisfied, iff $x=0$ or $y=0$. So we have

$$\Omega = \{z\in\mathbb{C}:\ \mathrm{Re} z = 0 \text{ or } \mathrm{Im} z = 0\}.$$

Of course the interior of Ω is empty.

Exercise G3 (Path integrals)

Consider the vector field

$$\mathbb{R}^2 \ni (x,y) \to F(x,y) := \frac{1}{(x^2+y^2+1)^2}\begin{pmatrix} -x^2+y^2+1 \\ -2xy \end{pmatrix} \in \mathbb{R}^2.$$

Determine $\int_{\gamma_1} F ds$ and $\int_{\gamma_2} F ds$ for the paths $\gamma_1 : [-1,1] \to \mathbb{R}^2$ and $\gamma_2 : [0,\pi] \to \mathbb{R}^2$ given by

$$\gamma_1(t) := \begin{pmatrix} -t \\ 0 \end{pmatrix} \qquad \text{and} \qquad \gamma_2(t) := \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}.$$

Hints for solution: We write in this hint for solution $x \cdot y := \langle x, y \rangle$ for vectors x and y and we write vectors as row vectors.
Let $W := \gamma_1$:

$$W : [-1,1] \to \mathbb{R}^2, \qquad W(t) = (-t, 0)$$

and let $Z := \gamma_2$:

$$Z : [0,\pi] \to \mathbb{R}^2, \qquad Z(t) = (\cos(t), \sin(t)).$$

We get

$$\int_W F dt = \int_{-1}^{1} F(W(t)) \cdot \dot{W}(t) dt = \int_{-1}^{1} \Big(\frac{-t^2+1}{(t^2+1)^2}, 0\Big) \cdot (-1,0) dt =$$

$$= \int_{-1}^{1} \frac{t^2-1}{(t^2+1)^2} dt = -\frac{t}{t^2+1}\Big|_{-1}^{1} = -1,$$

$$\int_Z F dt = \int_0^{\pi} F(Z(t)) \cdot \dot{Z}(t) dt = \int_0^{\pi} \Big(\frac{-\cos(2t)+1}{4}, \frac{-\sin(2t)}{4}\Big) \cdot (-\sin(t), \cos(t)) dt =$$

$$= \int_0^{\pi} \frac{\cos(2t)\sin(t) - \sin(t) - \sin(2t)\cos(t)}{4} dt =$$

$$= \int_0^{\pi} \frac{\sin(-2t)\cos(t) + \cos(-2t)\sin(t) - \sin(t)}{4} dt = \int_0^{\pi} \frac{\sin(-2t+t) - \sin(t)}{4} dt =$$

$$= \int_0^{\pi} \frac{-\sin(t)}{2} dt = \frac{\cos(t)}{2}\Big|_0^{\pi} = -1.$$

Exercise G4 (Elementary properties of the path integral)

Let $F,\ G : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable vector fields. Further let $\gamma, \gamma_1 : [a,b] \to \mathbb{R}^n$ and $\gamma_2 : [b,c] \to \mathbb{R}^n$ be continuously differentiable paths. Show that the path integral has the following properties:

(a) $\int_\gamma \lambda F + \mu G ds = \lambda \int_\gamma F ds + \mu \int_\gamma G ds$.

(b) $\int_{\gamma_1+\gamma_2} F ds = \int_{\gamma_1} F ds + \int_{\gamma_2} F ds$.

(c) If $\varphi : [\alpha, \beta] \to [a,b]$ is a diffeomorphism with $\varphi'(t) > 0$ then $\int_\gamma F ds = \int_{\gamma\circ\varphi} F ds$.

Interprete part (c) in the special case of a "vector field" $F : \mathbb{R} \supseteq [a,b] \to \mathbb{R}$ and the path $\gamma : [a,b] \to \mathbb{R}$, $\gamma(t) = t$.

Hints for solution: Standard Calculations. Part (c) is a multi dimensional version of integration by substitution.

Exercise G5 (Rotation of a vector field and a two dimensional version of Stoke's theorem)

Let $\Omega \subseteq \mathbb{R}^2$ be an open subset and $f : \mathbb{R}^2 \supseteq \Omega \to \mathbb{R}^2$ be a continuously differentiable vector field. Further let $v \in \Omega$ be an arbitrary point and $\varepsilon > 0$. Assume that the closed square with side length ε and center v is contained in Ω and let γ be the canonical parametrisation of the boundary of this square, i. e. it is counterclockwisely orientated.

(a) Prove that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_\gamma f\, ds = \operatorname{rot}(f)(v),$$

where $\operatorname{rot}(f)(x,y) := \frac{\partial f_2}{\partial x}(x,y) - \frac{\partial f_1}{\partial y}(x,y)$ defines the rotation of f.

(b) Prove Stoke's theorem in the two dimensional case:
Let $f : \mathbb{R}^2 \supseteq \Omega \to \mathbb{R}^2$ be a continuously differentiable vector field and $R := [a,b] \times [c,d]$ be a rectangle with $R \subseteq \Omega$. If γ is the canonical parametrisation of the boundary of R then the following equation holds:

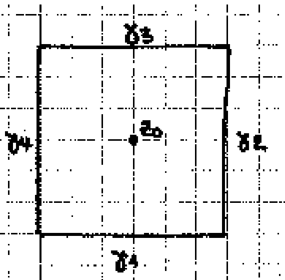
$$\int_\gamma f\, ds = \int_c^d \int_a^b \operatorname{rot}(f)(x,y)\,dx\,dy.$$

Hint: Use Fubini's theorem.

Hints for solution:

This is purely analysis:

(G2) •



$$\gamma_1 : t \mapsto \begin{pmatrix} x_0 - \varepsilon/2 \\ y_0 - \varepsilon/2 \end{pmatrix} + t \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix} \quad t \in [0,1]$$

$$\gamma_2 : t \mapsto \begin{pmatrix} x_0 + \varepsilon/2 \\ y_0 - \varepsilon/2 \end{pmatrix} + t \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} \quad t \in [0,1]$$

$$\gamma_3 : t \mapsto \begin{pmatrix} x_0 + \varepsilon/2 \\ y_0 + \varepsilon/2 \end{pmatrix} + t \begin{pmatrix} -\varepsilon \\ 0 \end{pmatrix} \quad t \in [0,1]$$

$$\gamma_4 : t \mapsto \begin{pmatrix} x_0 - \varepsilon/2 \\ y_0 + \varepsilon/2 \end{pmatrix} + t \begin{pmatrix} 0 \\ -\varepsilon \end{pmatrix} \quad t \in [0,1]$$

$$\int_\gamma f\,ds = \sum_{i=1}^{4} \int_{\gamma_i} f\,ds$$

$$= \int_0^1 f(\gamma_1(t)) \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix} dt + \int_0^1 f(\gamma_2(t)) \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} dt + \int_0^1 f(\gamma_3(t)) \begin{pmatrix} -\varepsilon \\ 0 \end{pmatrix} dt + \int_0^1 f(\gamma_4(t)) \begin{pmatrix} 0 \\ -\varepsilon \end{pmatrix} dt$$

$$= \int_0^1 f_1\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \varepsilon \begin{pmatrix} t - 1/2 \\ -1/2 \end{pmatrix}\right) \varepsilon - f_1\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \varepsilon \begin{pmatrix} t - 1/2 \\ -1/2 \end{pmatrix}\right) \varepsilon$$

$$+ f_2\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \varepsilon \begin{pmatrix} 1/2 \\ t - 1/2 \end{pmatrix}\right) \varepsilon - f_2\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \varepsilon \begin{pmatrix} 1/2 \\ t - 1/2 \end{pmatrix}\right) \varepsilon \; dt$$

$$\frac{1}{\varepsilon^2} \int_\gamma f\,ds = \frac{1}{\varepsilon} \int_0^1 f_1\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \varepsilon \begin{pmatrix} t - 1/2 \\ -1/2 \end{pmatrix}\right) - f_1\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \varepsilon \begin{pmatrix} t - 1/2 \\ -1/2 \end{pmatrix}\right)$$

$$+ f_2\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \varepsilon \begin{pmatrix} 1/2 \\ t - 1/2 \end{pmatrix}\right) - f_2\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \varepsilon \begin{pmatrix} 1/2 \\ t - 1/2 \end{pmatrix}\right) dt$$

Nullen einschieben

$$= \int_0^1 \frac{f_1\left(z_0 + \varepsilon \begin{pmatrix} t - 1/2 \\ -1/2 \end{pmatrix}\right) - f(z_0)}{\varepsilon - 0} + \frac{f(z_0) - f\left(z_0 - \varepsilon \begin{pmatrix} t - 1/2 \\ -1/2 \end{pmatrix}\right)}{0 - (-\varepsilon)}$$

$$+ \frac{f_2\left(z_0 + \varepsilon \begin{pmatrix} 1/2 \\ t - 1/2 \end{pmatrix}\right) - f(z_0)}{\varepsilon - 0} + \frac{f(z_0) - f_2\left(z_0 - \varepsilon \begin{pmatrix} 1/2 \\ t - 1/2 \end{pmatrix}\right)}{0 - (-\varepsilon)} \; dt$$

Richt'abl. erkennen

$$\lim_{\varepsilon \to 0} \int_\gamma f\,ds = \int_0^1 \left(\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}\right)(z_0) \cdot \begin{pmatrix} t - 1/2 \\ -1/2 \end{pmatrix} + \left(\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}\right)(z_0) \cdot \begin{pmatrix} t - 1/2 \\ -1/2 \end{pmatrix}$$

$$+ \left(\frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial y}\right)(z_0) \begin{pmatrix} 1/2 \\ t - 1/2 \end{pmatrix} + \left(\frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial y}\right)(z_0) \begin{pmatrix} 1/2 \\ t - 1/2 \end{pmatrix} \; dt$$

$$= 2 \int_0^1 \frac{\partial f_1}{\partial x}(z_0)\left(t - \tfrac{1}{2}\right) - \frac{\partial f_1}{\partial y}(z_0) \cdot \frac{1}{2} + \frac{\partial f_2}{\partial x}(z_0) \cdot \frac{1}{2} + \frac{\partial f_2}{\partial y}(z_0)\left(t - \tfrac{1}{2}\right) dt$$

$$= \not{2} \cdot \left[\frac{\partial f_1}{\partial x}(z_0)\left(\frac{t^2}{\not{2}} - \frac{t}{\not{2}}\right) - \frac{\partial f_1}{\partial y}(z_0)\frac{t}{\not{2}} + \frac{\partial f_2}{\partial x}(z_0)\frac{t}{\not{2}} + \frac{\partial f_2}{\partial y}(z_0)\left(\frac{t^2}{\not{2}} - \frac{t}{\not{2}}\right) \right]_0^1$$

$$= -\frac{\partial f_1}{\partial y}(z_0) + \frac{\partial f_2}{\partial x}(z_0)$$

(G3) Zunächst stellen wir fest, daß R kompakt ist, f und seine partiellen Ableitungen stetig sind, wir können also den Satz von Fubini anwenden (*)

$$\int_c^d \int_a^b \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\, dx\, dy \overset{(*)}{=} \int_c^d \int_a^b \frac{\partial f_2}{\partial x}\, dx\, dy - \int_a^b \int_c^d \frac{\partial f_1}{\partial y}\, dy\, dx$$

$$= \int_c^d f_2(b,y) - f_2(a,y)\, dy - \int_a^b f_1(x,d) - f_1(x,c)\, dx = (A)$$

Wir setzen

$$\left.\begin{aligned} \gamma_1(t) &= \binom{b}{0} + t\binom{0}{1},\ t \in [c,d] \\ \gamma_2(t) &= \binom{a}{0} + t\binom{0}{1},\ t \in [c,d] \end{aligned}\right\}\ \gamma_i'(t) = \binom{0}{1}$$

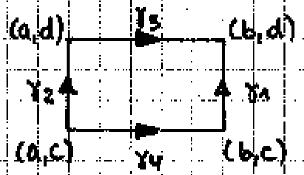
$$\left.\begin{aligned} \gamma_3(t) &= \binom{0}{d} + t\binom{1}{0},\ t \in [a,b] \\ \gamma_4(t) &= \binom{0}{c} + t\binom{1}{0},\ t \in [a,b] \end{aligned}\right\}\ \gamma_i'(t) = \binom{1}{0}$$

und erhalten somit

$$(A) = \int_c^d f(\gamma_1(t))\gamma_1'(t)\, dt - \int_c^d f(\gamma_2(t))\gamma_2'(t)\, dt$$

$$- \int_a^b f(\gamma_3(t))\gamma_3'(t)\, dt + \int_a^b f(\gamma_4(t))\gamma_4'(t)\, dt$$

$$= \int_{\gamma_1} f\, ds - \int_{\gamma_2} f\, ds - \int_{\gamma_3} f\, ds + \int_{\gamma_4} f\, ds$$



$$= \int_{\gamma} f\, ds$$

Homework

Exercise H1 (Connectedness and path-connectedness) (1 point)

Let (X,d) a metric space. The space X is called *connected*, if the only subsets of X which are both open and closed are X and the empty set.

(a) Prove that the following conditions are equivalent:
   (i) The space X is connected.
   (ii) If $X = A \cup B$ for open sets A and B with $A \cap B = \emptyset$, then $A = \emptyset$ or $B = \emptyset$.
   (iii) If $X = A \cup B$ for closed sets A and B with $A \cap B = \emptyset$, then $A = \emptyset$ or $B = \emptyset$.
   (iv) Every continuous function $f : X \to \{0, 1\}$ is constant.

(b) Is there a metric on $\mathbb{R}$ such that $(\mathbb{R}, d)$ is disconnected, i. e. not connected? Prove your claim.

(c) Show that every path connected metric space is connected.

(d) Let
$$\Gamma := \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right)^T : 0 < x \leq 1 \right\} \subseteq \mathbb{R}^2.$$
Define $X := \overline{\Gamma}$ where the closure is taken in the natural metric. Then (X, d) is a metric space with $d(x, y) := \left\| x - y \right\|_2$. Sketch the set X and show that X is connected but not path connected.

Hints for solution:

(a) (i) $\Rightarrow$ (ii): Let $X = A \cup B$ with $A \cap B = \emptyset$ and A open and B open. Then $B = A^C$ is a closed set. So A is open and closed. By (i) we have $A \in \{\emptyset, X\}$ so $B \in \{\emptyset, X\}$.
(ii) $\Leftrightarrow$ (iii) Cause $A = B^C$ and $B = A^C$ these conditions are trivially equivalent.
(ii) $\Rightarrow$ (i) Let $A \subseteq X$ be open and closed. Then A^C is open and closed, too. Further we have $X = A \cup A^C$ with $A \cap A^C = \emptyset$. By (ii) we have $A \in \{\emptyset, X\}$, so there is no nontrivial open and closed subset of X.
(i) $\Rightarrow$ (iv): Assume there is a surjective $f : X \to \{0, 1\}$ then $A = f^{-1}(\{0\})$ is open and closed and not empty and $B = f^{-1}(\{1\})$ is open and closed and not empty since $\{0, 1\}$ is discrete. So X is disconnected.
(iv) $\Rightarrow$ (iii): Assume, $X = A \cup B$ with $A \cap B = \emptyset$ and A closed and B closed. Define
$$f(x) = \begin{cases} 1 & x \in A \\ 0 & x \in B. \end{cases}$$
Since the preimage of closed sets in $\{0, 1\}$ is closed in X, the function f is continuous and not constant.

(b) Choose the discrete metric on $\mathbb{R}$, i. e.
$$d(x, y) = \begin{cases} 0 & x = y \\ 42 & x \neq y \end{cases}.$$
Then the set $A := \{42\}$ is closed and we have $A = K_\varepsilon(42)$ with $\varepsilon = \frac{1}{42}$. It follows that A is open. Of course, every subset $B \subset \mathbb{R}$ in the discrete topology is open and closed.

(c) Assume (X,d) to be path connected and let $A \subseteq X$ be open and closed. Define a function $f : X \to \{0,1\}$ by

$$f(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

This function is continuous since every preimage of an open set $B \subset \{0,1\}$ (of course with discrete topology, i. e. discrete metric) is open in X. Let $a \in A$ and $b \in X$ be arbitrary. Then by path connectedness there is a continuous path $\gamma[0,1] \to X$ with $\gamma(0) = a$ and $\gamma(1) = b$. Then of course the function $f \circ \gamma : [0,1] \to \{0,1\}$ is a continuous path, too. Since this path must be constant, we have $f(\gamma(1)) = f(\gamma(0)) = 1$. So we have $b = \gamma(1) \in A$ and since $b \in X$ was arbitrary we conclude $A = X$. So X is connected.

(d) The set X is the union of the graph of $\varphi :]0,1] \to \mathbb{R}$, $\varphi(x) = \sin(1/x)$ and the strip of accumulation points $S := \{(0,y)^T : -1 \leq y \leq 1\}$. This is easy to show by using the continuity of φ on $]0,1]$ and the fact, every $t \in [-1,1]$ is a limit of the image of a nullsequence under the map φ.
We have $X = \Gamma \cup S$. This is a disjoint union of path connected subsets. Assume $X = A \cup B$ as a disjoint union of both open and closed sets A and B. Then there is a continuous function $g : X \to \{0,1\}$ with $g(a) = 0$ for all $a \in A$ and $g(b) = 1$ for all $b \in B$. We have oBdA $S \subseteq A$. Since S is not open, there is a subset $C \subseteq X$ with $A = S \cup C$ and $S \cap C = \emptyset$. But for every $c \in C$ and every $x \in \Gamma$ there is a continuous path $\gamma : [0,1] \to X$ with $\gamma(0) = c$ and $\gamma(1) = x$. We conclude $A = X$ and $B = \emptyset$, so X is connected.
We have to show that X is not path connected. Choose $x = (1, \sin(1))$ and assume there is for a $s \in S$ a path γ with $\gamma(0) = x$ and $\gamma(1) = s$. Since S is closed there is a smallest number $a \in [0,1]$ with $\gamma(t) \in \Gamma$ for $0 \leq t < a$ and $s := \gamma(a) \in S$. Let p be the projection map $p : X \to \mathbb{R}$, $p(x,y) := x$. This map is continuous. Define for $n > 0$ the set $K_n := K_{\frac{1}{n}}(s)$. We have $\Gamma \cap K_n \neq \emptyset$ and $p(K_n) \subseteq [0, \frac{1}{n}]$. Since $\gamma : [0,a] \to X$ is a continuous path from x to s and the map $p \circ \gamma$ is continuous, we have by the intermediate value theorem $p \circ \gamma([0,a]) = [0,1]$: The preimage of K_n under γ^{-1} is an open subset of $[0,1]$. This means there is for every $n > 0$ a point $t \in [0,a[$ with $\gamma(t) \in K_n$ and we follow $\Gamma \subseteq \gamma([0,a])$.
Now there a different ways to prove that this path can't exist. The first way: $\gamma : [0,a] \to X$ is uniformly continuously since $[0,a]$ is compact. This means there is a $\delta > 0$ with $|s-t| \leq \delta \Rightarrow \left\|\gamma(s) - \gamma(t)\right\| \leq \frac{1}{2}$. But for $s_n = \frac{2}{n\pi}$ and $t_n = \frac{2}{(n+1)\pi}$ one can choose $n > 0$ large enough to get a contradiction.
The second way: The image $\gamma([0,a])$ is compact expecially closed in X. But we have $\gamma([0,a]) \cap S = \{s\}$ and $X = \overline{\Gamma} \subseteq \overline{\gamma([0,a])} = \gamma([0,a]) \neq X$ is a contradiction.

Exercise H2 (Curves, path length and rectifiability I) (1 point)

We first introduce some notation. A *partition* Z of $[0,1]$ is given by a finite ordered subset $Z = \{t_0, ..., t_n\}$ with $0 = t_0 < t_1 < t_2 < ... < t_n = 1$. For simplicity we write $Z = \{t_0, ..., t_n\}$.
Let $\gamma : [0,1] \to \mathbb{R}^n$ be a continuously differentiable path and Z a Partition of $[0,1]$. We define piecewise a new path $\gamma_Z : [0,1] \to \mathbb{R}^n$: For $t \in [t_n, t_{n+1}]$ we set

$$\gamma_Z(t) := \frac{t_{n+1} - t}{t_{n+1} - t_n} \cdot \gamma(t_n) + \frac{t - t_n}{t_{n+1} - t_n} \cdot \gamma(t_{n+1}).$$

Then γ_Z approximates γ by a polygon.
To understand this we consider an example: Let $\gamma : [0,1] \to \mathbb{R}^2$ defined by

$$\gamma(t) := \begin{pmatrix} \cos(\pi \cdot t) \\ \sin(\pi \cdot t) \end{pmatrix}.$$

Let Z_n be the partitions $\left\{0, \frac{1}{n}, \frac{2}{n}, ..., 1\right\}$.

(a) Visualise the path γ and the paths γ_{Z_2} and γ_{Z_3}.

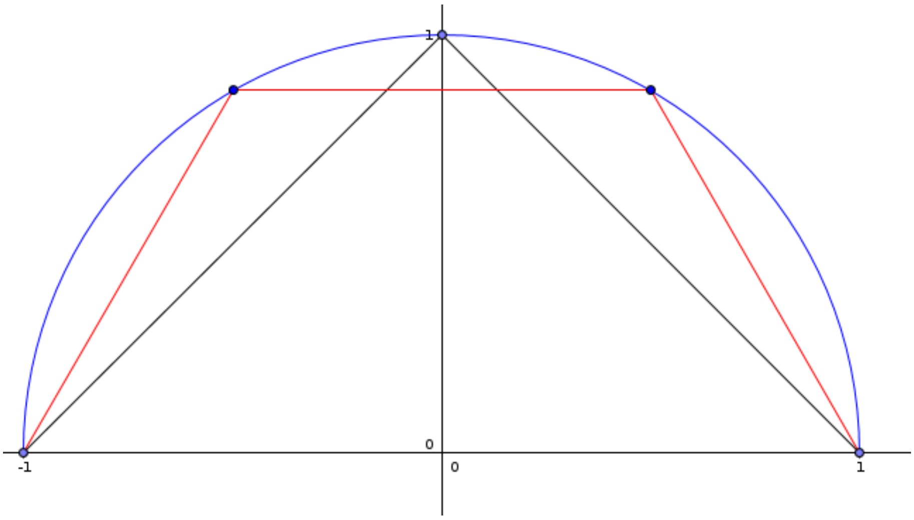
(b) Determine the length $L(\gamma)$ and $L(\gamma_{Z_n})$ for each $n \in \mathbb{N} \setminus \{0\}$.

(c) Show that $L(\gamma) = \lim_{n\to\infty} L(\gamma_{Z_n})$.

Remark: Let $\gamma : [0,1] \to \mathbb{R}^n$ a path which is continuously differentiable except in finitely many points, then the length of γ is defined by

$$L(\gamma) := \int_0^1 \left\|\gamma'(t)\right\| dt.$$

Hints for solution:



(a)

(b) We calculate:

$$\begin{aligned} L(\gamma) &= \int_0^1 \left\|\gamma'(t)\right\| dt = \int_0^1 \pi \sqrt{\cos^2(\pi \cdot t) + \sin^2(\pi \cdot t)} dt \\ &= \pi. \end{aligned}$$

Let $n > 0$ then we have with $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$ and $x = \frac{k+1}{n}\pi$ and $y = \frac{k}{n}\pi$:

$$\begin{aligned}
L(\gamma_{Z_n}) &= \int_0^1 \left\|\gamma_{Z_n}(t)\right\| dt = \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} n \cdot \left\|\gamma\left(\frac{k+1}{n}\pi\right) - \gamma\left(\frac{k}{n}\pi\right)\right\| dt \\
&= \sum_{k=0}^{n-1} \left\| \begin{pmatrix} \cos\left(\frac{k+1}{n}\pi\right) - \cos\left(\frac{k}{n}\pi\right) \\ \sin\left(\frac{k+1}{n}\pi\right) - \sin\left(\frac{k}{n}\pi\right) \end{pmatrix} \right\| \\
&= \sum_{k=0}^{n-1} \sqrt{2 - 2\left(\cos\left(\frac{k+1}{n}\pi\right) \cdot \cos\left(\frac{k}{n}\pi\right) + \sin\left(\frac{k+1}{n}\pi\right) \cdot \sin\left(\frac{k}{n}\pi\right)\right)}. \\
&= \sum_{k=0}^{n-1} \sqrt{2 - 2\cos\left(\frac{\pi}{n}\right)} \\
&= \sqrt{2} \cdot n \cdot \sqrt{1 - \cos\left(\frac{\pi}{n}\right)}.
\end{aligned}$$

(c) After this calculation we have to consider the limit of these numbers. Happy about L'Hospitals rule we get:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{2} \cdot n \cdot \sqrt{1 - \cos\left(\frac{\pi}{n}\right)} &= \sqrt{2} \cdot \lim_{n\to\infty} \frac{\sqrt{1 - \cos\left(\frac{\pi}{n}\right)}}{\frac{1}{n}} \\
&= \sqrt{2} \cdot \lim_{n\to\infty} \frac{-\sin\left(\frac{\pi}{n}\right) \cdot \frac{\pi}{n^2}}{-2 \cdot \sqrt{1 - \cos\left(\frac{\pi}{n}\right)} \cdot \frac{1}{n^2}} \\
&= \frac{\pi}{\sqrt{2}} \cdot \sqrt{\lim_{n\to\infty} \frac{1 - \cos^2\left(\frac{\pi}{n}\right)}{1 - \cos\left(\frac{\pi}{n}\right)}} \\
&= \frac{\pi}{\sqrt{2}} \cdot \sqrt{\lim_{n\to\infty} \frac{2\sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}} \\
&= \pi.
\end{aligned}$$

Exercise H3 (Curves, path length and rectifiability II) (1 point)

Let $\gamma : [0,1] \to \mathbb{R}^n$ be a path. We call γ rectifiable, if the following supremum exists:

$$l(\gamma) = \sup\{L(\gamma_Z) : \ Z \text{ is a partition of } [0,1]\}.$$

Let Z be a partition of $[0,1]$. We call a partition Z' of $[0,1]$ a refinement of Z, if $Z \subseteq Z'$ and write $Z \leq Z'$. The *mesh* $|Z|$ of a partition $Z = \{0 = t_0, t_1, ..., t_n = 1\}$ is defined by

$$|Z| := \max\{t_{k+1} - t_k : \ 0 \leq k \leq n-1\}.$$

(a) Show that for each refinement $Z \leq Z'$ one has $L(\gamma_Z) \leq L(\gamma_{Z'})$.

(b) Show that every continuously differentiable path is rectifiable with $l(\gamma) = L(\gamma)$.

Hints for solution:

(a) Assume one has a refinement Z' of Z. Then for each pair (s_i, s_{i+1}) consecutive elements of Z one has a finite chain $t_0^i < ... < t_n^i$ in Z' with $t_0^i = s_i$ and $t_n^i = s_{i+1}$. Witht the triangle inequality one gets

$$\left\|\gamma(s_{i+1}) - \gamma(s_i)\right\| = \left\|\sum_{j=0}^{n-1} \gamma(t_{j+1}^i) - \gamma(t_j^i)\right\| \leq \sum_{j=0}^{n-1} \left\|\gamma(t_{j+1}^i) - \gamma(t_j^i)\right\|.$$

Summarising over all consecutive pairs (s_i, s_{i+1}) proves the claim.

(b) Let $\varepsilon > 0$. We have to show $\left|l(z) - \int_0^1 \left\|\gamma'(t)\right\| dt\right| \leq \varepsilon$.

Choose a partition Z_0 with $|l(\gamma) - L(\gamma_{Z_0})| \leq \frac{\varepsilon}{2}$. Of course $t \to \gamma'(t)$ is uniformly continuous and so is $t \to \left\|\gamma'(t)\right\|$. So we can find a $\delta > 0$ with $|s - t| < \delta \Rightarrow \left\|\gamma'(s) - \gamma'(t)\right\| < \frac{\varepsilon}{2n}$. Let $Z \geq Z_0$ with $|Z| < \delta$. By (a) we have $|l(\gamma) - L(\gamma_Z)| \leq \frac{\varepsilon}{2}$. Let $m \in \mathbb{N}$ the number of elements of Z.

We conclude using the mean value theorems for differentiation and integration:

$$\begin{aligned}
\left|L(\gamma_Z) - \int_0^1 \left\|\gamma'(t)\right\| dt\right| &= \left|\sum_{j=0}^{m-1} \left\|\gamma(t_{j+1}) - \gamma(t_j)\right\| - \int_{t_j}^{t_{j+1}} \left\|\gamma'(t)\right\| dt\right|. \\
&= \left|\sum_{j=0}^{m-1} \left\|\begin{pmatrix} \gamma_1'(\tau_1^j) \\ \vdots \\ \gamma_n'(\tau_n^j) \end{pmatrix}\right\| (t_{j+1} - t_j) - \left\|\gamma'(\tau^j)\right\| (t_{j+1} - t_j)\right|
\end{aligned}$$

with $\tau^j, \tau_1^j, ..., \tau_n^j \in [t_j, t_{j+1}]$. Since $|\gamma_i'(s) - \gamma_i'(t)| \leq \left\|\gamma'(s) - \gamma'(t)\right\|| < \frac{\varepsilon}{2n}$ we conclude

$$\begin{aligned}
\left|L(\gamma_Z) - \int_0^1 \left\|\gamma'(t)\right\| dt\right| &= \left|\sum_{j=0}^{m-1} \left\|\begin{pmatrix} \gamma_1'(\tau_1^j) \\ \vdots \\ \gamma_n'(\tau_n^j) \end{pmatrix}\right\| (t_{j+1} - t_j) - \left\|\gamma'(\tau^j)\right\| (t_{j+1} - t_j)\right| \\
&\leq \sum_{j=0}^{m-1} \left|\left\|\begin{pmatrix} \gamma_1'(\tau_1^j) \\ \vdots \\ \gamma_n'(\tau_n^j) \end{pmatrix}\right\| (t_{j+1} - t_j) - \left\|\gamma'(\tau^j)\right\| (t_{j+1} - t_j)\right| \\
&\leq n \cdot \frac{\varepsilon}{2n} = \frac{\varepsilon}{2}
\end{aligned}$$

This means

$$|l(\gamma) - L(\gamma)| \leq |l(\gamma) - L(\gamma_Z)| + |L(\gamma_Z) - L(\gamma)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof.