Analysis III – Complex Analysis
Hints for solution for the 1. Exercise Sheet

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Groupwork

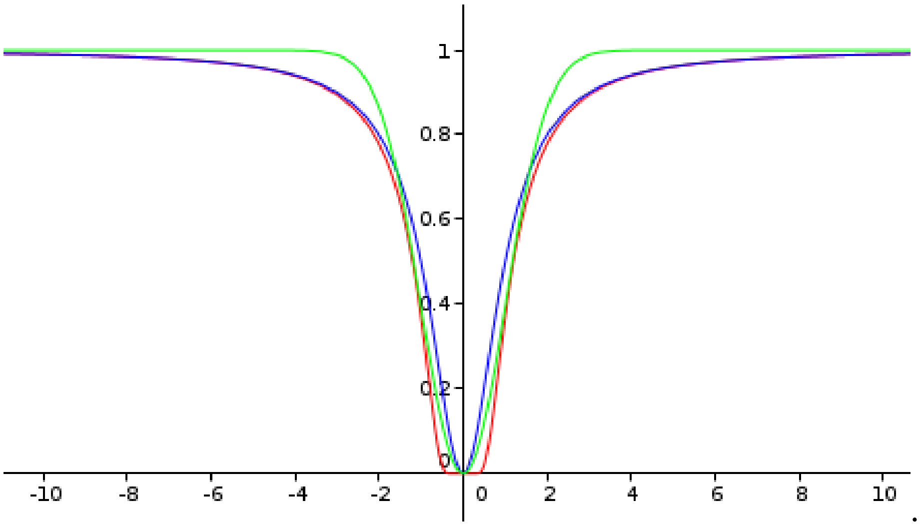
Exercise G1 (Power series of real functions)

We consider the following functions which are defined on the whole real axis:

$$f_1(x) := \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0, \end{cases} \qquad f_2(x) := \frac{x^2}{1+x^2}, \qquad f_3(x) := 1 - e^{-\frac{x^2}{2}}.$$

Sketch the graphs of these functions and expand them in $x_0 = 0$ into a Taylor series. Determine for each Taylor series the greatest open subset $U \subset \mathbb{R}$ such that the series represents the function.

Hints for solution: The graphs of these functions look like very simmilar:



The Taylor series of these functions behave in opposite completely different:

$$T_1(x) = 0, \qquad T_2(x) = \sum_{n=0}^{\infty} (-1)^n \cdot x^{2n+2}, \qquad T_3(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k \cdot k!} \cdot x^{2k}.$$

The maximal open subset for T_1 is $\emptyset$, because only in $x_0 = 0$ it represents f_1. The maximal open subset for T_2 is $]-1,1[$ and for T_3 is $\mathbb{R}$.

Exercise G2 (Complex functions and real vector fields)

We already know that $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$ as a real vector space with the canonical $\mathbb{R}$-Basis $\{1, i\}$. In this way we identify the complex numer $z = a + bi$ with the vector $\begin{pmatrix} a \\ b \end{pmatrix}$. For this exercise we call this identification the *canonical identification of* $\mathbb{C}$ *with* $\mathbb{R}^2$.

Now we consider the complex polynomial $f : \mathbb{C} \to \mathbb{C}$ with $f(z) := z^2 + 1$.

(a) Show that f is complex differentiable in the following sense: For each complex number $z \in \mathbb{C}$ the limit

$$f'(z) := \lim_{\omega \to 0} \frac{f(z+\omega) - f(z)}{\omega}$$

exists. Calculate $f'(z)$ explicitely.

(b) We define the real vector field

$$F(x, y) := \begin{pmatrix} \mathrm{Re}(f(x + y \cdot i)) \\ \mathrm{Im}(f(x + y \cdot i)) \end{pmatrix}.$$

Show that this vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ is everywhere differentiable and calculate the Jacobian.

(c) Is there some remarkable relation of the Jacobian $J_F(x, y)$ and the value of $f'(x + yi)$?
Hint: Any complex linear funcion $T : \mathbb{C} \to \mathbb{C}$ is of course a real linear function.

Hints for solution:

(a) Exactly as in the real case one gets:

$$\begin{aligned} f'(z) &= \lim_{\omega \to 0} \frac{f(z+\omega) - f(z)}{\omega} = \lim_{\omega \to 0} \frac{z^2 + 2z\omega + \omega^2 + 1 - z^2 - 1}{\omega} \\ &= \lim_{\omega \to 0} \frac{(2z+\omega)\omega}{\omega} = \lim_{\omega \to 0} 2z + \omega = 2z. \end{aligned}$$

So the function f is complex differentiable with derivative $f'(z) = 2z$.

(b) We calculate

$$\begin{aligned} F(x, y) &= \begin{pmatrix} \mathrm{Re}(f(x + y \cdot i)) \\ \mathrm{Im}(f(x + y \cdot i)) \end{pmatrix} = \begin{pmatrix} \mathrm{Re}(x^2 - y^2 + 1 + 2xyi) \\ \mathrm{Im}(x^2 - y^2 + 1 + 2xyi) \end{pmatrix} \\ &= \begin{pmatrix} x^2 - y^2 + 1 \\ 2xy \end{pmatrix}. \end{aligned}$$

So we get

$$J_F(x, y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}.$$

(c) The map $\omega \to 2z\cdot\omega$ is a $\mathbb{C}$-linear map $\mathbb{C} \to \mathbb{C}$. Of course it is $\mathbb{R}$-linear too, so it is represented by a 2×2 matrix under canonical representation of $\mathbb{C}$ with $\mathbb{R}^2$. Let $z = x + yi$, $\omega = a + bi$ and $T_z(x, y)$ the $\mathbb{R}$-linear interpretation of $M_{2z}(\omega) := 2z \cdot \omega$. We get

$$\begin{aligned} T_z(x, y) &= 2 \cdot \begin{pmatrix} \mathrm{Re}((x + yi)(a + bi)) \\ \mathrm{Im}((x + yi)(a + bi)) \end{pmatrix} = 2 \cdot \begin{pmatrix} \mathrm{Re}(ax - by + (ay + bx)i) \\ \mathrm{Im}(ax - by + (ay + bx)i) \end{pmatrix} \\ &= 2 \cdot \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}. \end{aligned}$$

So $J_F(x, y) : \mathbb{R}^2 \to \mathbb{R}^2$ is the real linear vector field corresponding to $\omega \to f'(x + yi) \cdot \omega$.

Exercise G3 (Fields, matrices and complex numbers)

Let $\mathbb{K}$ be a field and let $\lambda \in \mathbb{K}$ be a number which has no square root in $\mathbb{K}$, i. e. there is no element $\mu \in \mathbb{K}$ with $\mu^2 = \lambda$.

Let $M_2(\mathbb{K})$ be the set of all 2×2 matrices with entries in $\mathbb{K}$. In this exercise we consider the subset

$$\mathbb{L} := \left\{ \begin{pmatrix} a & \lambda \cdot b \\ b & a \end{pmatrix}, \quad a, b \in \mathbb{K} \right\} \subseteq M_2(\mathbb{K}).$$

(a) Show that $\mathbb{L}$ is a field with the usual matrix addition and matrix multiplication. Assure yourself that

$$\mathbb{K} \ni x \to \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} =: x \cdot \mathbb{1} \in \mathbb{L}$$

defines an injective field homomorphism.

Hint: You may use your knowledge of matrices over fields to avoid proving every axiom for a field.

(b) in which way is $l := \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}$ special?

(c) What can you say about the eigenvalues of $a \cdot \mathbb{1} + b \cdot l$?

(d) Find a subset of $M_2(\mathbb{R})$ which is isomorphic to $\mathbb{C}$.

(e) Is there a field with 9 elements?

Hints for solution:

(a) It's a very easy calculation to prove that $\mathbb{L}$ is closed under addition and multiplication. Further it's very easy to show that the multiplication on $\mathbb{L}$ is commutative. So it's only necessary to prove every element in $\mathbb{L} \setminus \{0\}$ is invertible and the inverse is again in $\mathbb{L}$. Invertibility follows directly, because the determinant of such an element is $a^2 - \lambda b^2$ – it's a number in $\mathbb{K}$ which can't be zero by assumption on λ. The inverse is an element of $\mathbb{L}$ again which can be seen easily by calculation or formulas from linear algebra.

The map $\mathbb{K} \ni x \to x \cdot \mathbb{1}$ is obviously $\mathbb{K}$-linear, multiplicative and injective. Further $\mathbb{1}$ is the neutral element in $M_2(\mathbb{K})$, so the image of $1 \in \mathbb{K}$ is $\mathbb{1} \in \mathbb{L}$. So it's an injective field homomorphism.

(b) In $\mathbb{L}$ the following equation holds:

$$\begin{aligned} l^2 &= \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}. \end{aligned}$$

So the equation $X^2 - \lambda = 0$ has at least one solution l in $\mathbb{L}$. In fact $-l$ is a solution, too.

(c) If $b \neq 0$ holds this element has no eigenvalues since the polynomial $p(X) = (a - X)^2 - \lambda b^2$ has no roots in $\mathbb{K}$. If $b = 0$, one easily sees the eigenvalue: a. Of course, the eigenspace is 2-dimensional in this case.

(d) Choose $\lambda = -1$ and write down $\mathbb{L}$.

(e) In $\mathbb{Z}_3$ which is a field there is no root for $\overline{2}$. So we can choose $\lambda = \overline{2}$ and write down $\mathbb{L}$. Sadly: In a finite field of characteristic 2 the construction above in this exercise is not applicable (why?).

Exercise G4 (Visualisation of complex functions)

Consider the complex polynomial $f : \mathbb{C} \to \mathbb{C}$ with $f(z) = z^2$ and the following subset M of $\mathbb{C}$:

$$M := \{z \in \mathbb{C} : \ 0 \le \mathrm{Re}(z) \le 1, \ 0 \le \mathrm{Im}(z) \le 1\}.$$

(a) Is M open, closed, bounded, compact, convex?

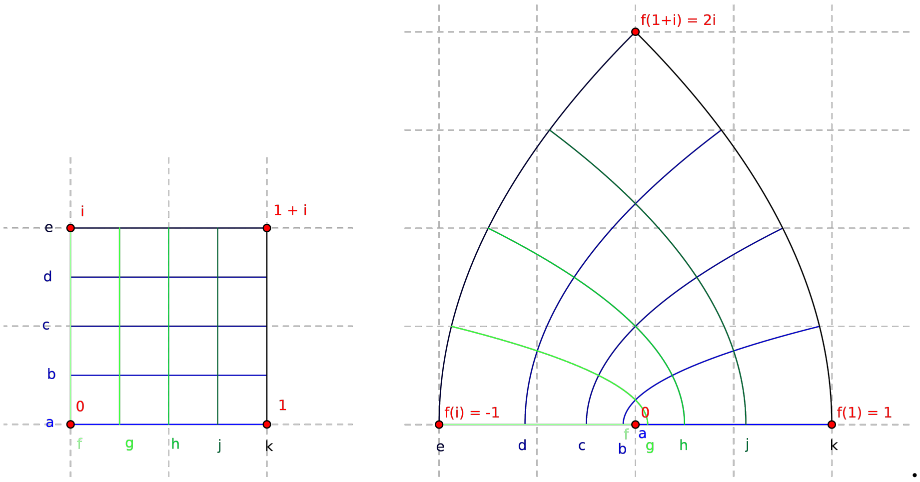
(b) Calculate the image $f(M)$ and visualize the action of f by laying a grid into M, paramterizing grid lines by paths and calculating the image under f of these paths. Draw them into a draft and look on the angles of intersecting image paths. Looks something particular?

(c) What is the image of the half disk $\{z \in \mathbb{C} : \mathrm{Im}(z) \ge 0 \text{ and } |z| < 1\}$?

Hints for solution:

(a) Of course M is not open. The answer of the other questions is yes.

(b) Look:



Except in $z = 0$, all images of rectangular angles look like infinitesimally staying rectangular.

(c) The image is the hole unit disk $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$.

Remark: The images in this hints for solution are drawn by GeoGebra and converted by GIMP.

Homework

Exercise H1 (Curves and path length) (1 point)

Let $\gamma : [a,b] \to \mathbb{R}^n$ be a regular path which parameterises a curve $\Gamma \subseteq \mathbb{R}^n$. The arc length $s : [a,b] \to \mathbb{R}$ of γ is defined as follows:

$$s(t) := \int_a^t \left\|\gamma'(x)\right\| dx.$$

(a) Calculate $s(t)$ for the path $\gamma : [1,2] \to \mathbb{R}^3$ with $\gamma(t) := \begin{pmatrix} 2 \cdot t \\ t^2 \\ \ln(t) \end{pmatrix}$.

(b) Why do we assume the path beeing regular instead of continuously differentiable?

(c) Show that $s : [a,b] \to [0, l(\gamma)]$ is a diffeomorphism for a regular path. Use this for writing down a parameterisation $\phi : [0, l(\gamma)] \to \Gamma$ (The parameterisation by the arc length).

(d) Consider the curve $\Gamma := \{(x,y) \in \mathbb{R}^2 : y^3 - x^2 = 0\} \cap [-1,1] \times [-1,1]$. Is it possible to parameterise this curve continuously differentiable? Is it possible to parameterise this curve regularly? Prove your claim.

Hints for solution:

(a)

$$\begin{aligned} s(t) &= \int_1^t \left\|\gamma'(u)\right\| du = \int_1^t \sqrt{4 + 4u^2 + \frac{1}{u^2}} du \\ &= \int_1^t 2u + \frac{1}{u} du = t^2 + \ln(t) - 1. \end{aligned}$$

(b) If we don't assume γ to be regular we can't exclude that the path stops and move backward (The integrant is nonnegative!) which causes nonintuitive arc lengths in our oppinion.

(c) The function s is continuous and strict monotonically growing. So f is injective. By the intermediate value theorem it is surjective, too. The derivative of s is $\left\|\gamma'(t)\right\| > 0$. So the inverse map $s^{-1} : [0, l(\gamma)] \to [a,b]$ is differentiable, too. Since the inverse mapping is continuously differentiable, s is a diffeomorphism.
Define $\phi(t) := \gamma(s^{-1}(t))$ and one get's the demanded parametrisation.

(d) Yes: $\gamma(t) := \begin{pmatrix} t^3 \\ t^2 \end{pmatrix}$ is a continuously differentiable parametrisation of Γ.
There can't be a regular representation: Assume we have a continuously differentiable parameterisation $\gamma : [0,1]$ of Γ. We use γ_1 and γ_2 for the components of the path γ. Because $(-1,1) \in \Gamma$ and $(1,1) \in \Gamma$ there is by the mean value theorem for differentiable real functions a point $t_0 \in]0,1[$ with $\gamma_2'(t_0) = 0$.
Further the component functions are related by $\gamma_1(t)^2 = \gamma_2(t)^3$, so we get after differentiation

$$2\gamma_1(t) \cdot \gamma_1'(t) = 3\gamma_2(t)^2 \cdot \gamma_2'(t).$$

From this equations we get $\gamma_1(t_0) = \gamma_2(t_0) = 0$ or $\gamma_1'(t_0) = 0$. In the second case, γ is not a regular parametrisation. So we have to discuss the first case.
Assume that $\gamma_1'(t_0) \neq 0$. So this is true in some open neighbourhood of $t_0 \in]0,1[$ since γ_1' is continuous. In this neighbourhood $\gamma_1(t)$ is not zero for $t \neq t_0$, too – elsewhere by the mean value theorem we would have a zero of γ_1' in the neighbourhood, a contradicion. We name this neighbourhood by U and using $\gamma_1(t) \neq 0$ for $t \in U \setminus \{t_0\}$. We get on $U \setminus \{t_0\}$:

$$\gamma_1'(t) = \frac{3}{2} \cdot \frac{\gamma_2(t)^2 \cdot \gamma_2'(t)}{\gamma_1(t)}.$$

Using $|\gamma_1(t)^2| = |\gamma_2(t)^3|$ we get on $U \setminus \{t_0\}$:

$$|\gamma_1'(t)| = \frac{3}{2} \cdot \left|\gamma_2(t)\right|^{\frac{1}{2}} \cdot |\gamma_2'(t)|.$$

Taking $\lim_{t \to t_0}$ on both sides we get $\gamma_1'(t_0) = 0$ a contradiction. So there can't exist a regular representation of Γ.

Exercise H2 (A very important vector field) (1 point)

Consider the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ defined by $f(z) := \frac{1}{z}$.

(a) Calculate the real vector field $F : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ which describes after canconical representation of $\mathbb{R}^2$ and $\mathbb{C}$ the function f.

(b) Determine all points $(x, y) \in \mathbb{R}^2 \setminus \{0\}$ in which F is differentiable. For which points $(x, y) \in \mathbb{R}^2 \setminus \{0\}$ is the Jacobian $J_F(x, y)$ the action of a complex linear map?

Hints for solution:

(a) An easy calculation shows

$$F(x, y) = \begin{pmatrix} \frac{x}{x^2+y^2} \\ \frac{-y}{x^2+y^2} \end{pmatrix}.$$

(b) The Jacobian of F in $(x, y)^T$ is given by

$$J_F(x, y) = \frac{1}{(x^2 + y^2)^2} \begin{pmatrix} -x^2 + y^2 & -2xy \\ 2xy & -x^2 + y^2 \end{pmatrix}.$$

For every $(x, y)^T \in \mathbb{R}^2 \setminus \{0\}$ the Jacobian $J_F(x, y)$ corresponds to the action of the map $M_\omega : \mathbb{C} \to \mathbb{C},\ M_\omega z = \omega \cdot z$ for $\omega = -\frac{1}{(x+yi)^2}$. This shouldn't be a hard surprise.

Exercise H3 (Path connectedness) (1 point)

Let (X,d) be a metric space. We call a metric space *path connected* if for any two poins $x,y\in X$ there is a continuous path $\gamma:[0,1]\to X$ with $\gamma(0)=x$ and $\gamma(1)=y$. Show the following statements:

(a) Let (X,d) and $(Y,\tilde{d})$ be metric spaces and $f:X\to Y$ a surjective continuous map. Then Y is path connected if X is path connected.

(b) The set of all orthorgonal 2×2 matrices over $\mathbb{R}$ called $O_2(\mathbb{R})$ is not path connected. For this you can choose any norm on $M_2(\mathbb{R})$ to get a metric on $O_2(\mathbb{R})$: The result is independend of the chosen norm.
Hint: You can use that the coordinate evaluation maps $A\to A_{i,j}$ are continuous. By (a) it must be possible to find a path disconnected metric space (Y,d) and a surjective continuous map $f:O_2(\mathbb{R})\to Y$.

(c) Let (X,d) and $(Y,\tilde{d})$ be metric spaces and let $\varphi:X\to Y$ be a homeomorphism. Then X is path connected iff Y is path connected.
Remark: 'Iff' means if and only if. It's a common and often used abbreviation in mathematical literature.

(d) There is no homeomorphism $f:\mathbb{R}\to\mathbb{C}$ if $\mathbb{R}$ and $\mathbb{C}$ carry the natural metric induced by the absolute value $|\cdot|$.

(e) There is no isomorphism of fields $\varphi:\mathbb{R}\to\mathbb{C}$.

(f**) There is a bijection $\Phi:\mathbb{R}\to\mathbb{C}$.

Remark: In the last steps we see an interesting fact: The real numbers and the complex numbers are different fields, different metric spaces but as sets they are equal in some sense.

Hints for solution:

(a) Let $x,y\in Y$ be arbitrary. We find preimages $a,b\in X$ with $f(a)=x$ and $f(b)=y$. Since X is path connected there is a continuous path γ which starts in a and ends in b. Build $\tilde{\gamma}(t):=f(\gamma(t))$ and you get a path in Y starting in x and ending in y. So Y is path connected.

(b) There are various characterisations for a matrix beeing orthogonal: Algebraically this means $A\in O_2(\mathbb{R})$ iff $A^T\cdot A=\mathbb{1}$ and geometrically this means the rows of A builds an orthogonal basis of $\mathbb{R}^2$ and the columns as well. If we looks at the matrix entries we see the images of the maps $A\to A_{i,j}=A\to\left\langle A\cdot e_i,e_j\right\rangle$ are the real intervall $[-1,1]$ which is path connected and brings no obvious counterexample.
If one look at the orientation of the orthonogal basis consisting of the columns or rows of A which are the images on the standard orthonormal basis there are two different cases: Either the orientation is preserved or not (compared with the orientation of the stadard orhonormal basis). If one varies A continuously the column vectors varies continuously and so the orientation stays preserved or reflected. This can be proved elegantly using the determinant: The determinant of A is positive iff the orientation is preserved and negative in the other case. Further the determinant is a continuous function on $M_2(\mathbb{R})$ (Leibnitz formula for the determinant!) and the image of the orthogonal matrices is $\{-1,1\}\subseteq\mathbb{R}$. So there cannot exist a continuous path from

$$A:=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\qquad\text{to}\qquad \mathbb{1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

cause the determinant of A is -1 and the determinant of $\mathbb{1}$ is 1. So the space $O_2(\mathbb{R})$ can't be path connected.

(c) Use (a) for φ and its by definition continuous inverse map φ^{-1}.

(d) Assume there is a homeomorphism $f : \mathbb{R} \to \mathbb{C}$. Then there is a point $x \in \mathbb{R}$ with $f(x) = 0$. OBdA $x = 0$. If we restrict f to $\mathbb{R} \setminus \{0\}$ then we get a homeomorphism $g : \mathbb{R} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$. Since $\mathbb{C} \setminus \{0\}$ is path connected and since $\mathbb{R} \setminus \{0\}$ is not path connected the inverse map g^{-1} cant be surjective and continuous, a contradiction. So there can't exist such an homeomorphism f.

(e) Assume we have a field isomorphism $\varphi : \mathbb{R} \to \mathbb{C}$. Then because $\mathbb{R}$ is an ordered field there is an ordering on $\mathbb{C}$. Because $-1 < 0$ and $x^2 \geq 0$ in every ordered field one gets $0 \leq i^2 = -1 < 0$ a contradiction. So there can't exist such an isomorphism.
Another proof without using ordering: Assume there is a field isomorphism φ. Because the equation $X^2 + 1 = 0$ has a solution in $\mathbb{C}$ it has a solution in $\varphi^{-1}(\mathbb{C})$ because $0 = \varphi^{-1}(\lambda^2+1) = \left(\varphi^{-1}(\lambda)\right)^2 + 1$. But in $\mathbb{R}$ there is no number x with $x^2 = -1$ a contradiction. So there can't exist such an isomorphism.

(f**) It's not easy to give an explicite bijective map. One can use set theoretic theorems like the Cantor-Bernstein-Schröder theorem and only has to find an injective map $f : \mathbb{R} \to \mathbb{C}$ and an injective map $h : \mathbb{C} \to \mathbb{R}$. The first injection is easy to find. Instead of finding an injective map $h : \mathbb{C} \to \mathbb{R}$ it is enough to find a surjective map $g : \mathbb{R} \to \mathbb{C}$. This can be done as follows: Every real numer in $]0,1]$ has a unique representation $x = \sum_{k=1}^{\infty} x_k \cdot 2^{-k}$ where the sequence $(x_n)_{n\in\mathbb{N}}$ is not finally zero. Define

$$\phi(x) := \left(\sum_{k=1}^{\infty} x_{2k} \cdot 2^{-k}, \quad \sum_{k=1}^{\infty} x_{2k-1} \cdot 2^{-k} \right).$$

This is a surjective map $\phi :]0,1] \to [0,1] \times [0,1] \setminus \{(0,0)\}$. So there is of course a surjective map $[0,1] \to [0,1] \times [0,1]$. Now we can use a surjective map $\Phi : [0,1] \times [0,1] \to \mathbb{C}$ to get a surjective map $g : \mathbb{R} \to \mathbb{C}$.
Of course this is very unconstructive.
The Cantor-Bernstein-Schröder theorem states: If one has an injection $f : X \to Y$ and an injection $g : Y \to X$ then there is a bijection $\Phi : X \to Y$.