Solving Equations

Example:

```
> restart;
> x + y = 23;
```

$$x + y = 23 \qquad (1)$$

```
> eq1 := (x^3 - 2·x^2 + 23·x - 108 = 0);
```

$$eq1 := x^3 - 2\,x^2 + 23\,x - 108 = 0 \qquad (2)$$

```
> eq2 := (2·x + 4·y = 29/6);
```

$$eq2 := 2\,x + 4\,y = \frac{29}{6} \qquad (3)$$

```
> res := solve(eq1, x);
```

$$res := \frac{1}{3}\left(1259 + 3\sqrt{206634}\right)^{1/3} - \frac{65}{3\left(1259 + 3\sqrt{206634}\right)^{1/3}} + \frac{2}{3}, -\frac{1}{6}\left(1259 + 3\sqrt{206634}\right)^{1/3} + \frac{65}{6\left(1259 + 3\sqrt{206634}\right)^{1/3}} + \frac{2}{3} + \frac{1}{2}\,\mathrm{I}\sqrt{3}\left(\frac{1}{3}\left(1259 + 3\sqrt{206634}\right)^{1/3} + \frac{65}{3\left(1259 + 3\sqrt{206634}\right)^{1/3}}\right), -\frac{1}{6}\left(1259 + 3\sqrt{206634}\right)^{1/3} + \frac{65}{6\left(1259 + 3\sqrt{206634}\right)^{1/3}} + \frac{2}{3} - \frac{1}{2}\,\mathrm{I}\sqrt{3}\left(\frac{1}{3}\left(1259 + 3\sqrt{206634}\right)^{1/3} + \frac{65}{3\left(1259 + 3\sqrt{206634}\right)^{1/3}}\right) \qquad (4)$$

```
> fsolve(eq1, x);
```

$$3.692418864 \qquad (5)$$

```
> ?fsolve
> fsolve({eq1, eq2}, {x, y});
```

$$\{x = 3.692418864, y = -0.6378760987\} \qquad (6)$$

```
> solve({eq1, eq2}, {x, y});
```

$$\left\{x = \mathrm{RootOf}\left(\_Z^3 - 2\,\_Z^2 + 23\,\_Z - 108, label = \_L2\right), y = -\frac{1}{2}\,\mathrm{RootOf}\left(\_Z^3 - 2\,\_Z^2 + 23\,\_Z - 108, label = \_L2\right) + \frac{29}{24}\right\} \qquad (7)$$

```
> evalf(res);
```

$$3.692418863, -0.8462094323 + 5.341633545\,\mathrm{I}, -0.8462094323 - 5.341633545\,\mathrm{I} \qquad (8)$$

```
> evalf(res[1]);
```

$$3.692418863 \qquad (9)$$

```
> reslist := convert({res},'list'); # also possible: reslist := [res];
```

(10)

$$reslist := \left[\frac{1}{3} \left(1259 + 3\sqrt{206634}\right)^{1/3} - \frac{65}{3\left(1259 + 3\sqrt{206634}\right)^{1/3}} + \frac{2}{3}, -\frac{1}{6} \left(1259 + 3\sqrt{206634}\right)^{1/3} + \frac{65}{6\left(1259 + 3\sqrt{206634}\right)^{1/3}} + \frac{2}{3} - \frac{1}{2} \mathrm{I}\sqrt{3} \left(\frac{1}{3} \left(1259 + 3\sqrt{206634}\right)^{1/3} + \frac{65}{3\left(1259 + 3\sqrt{206634}\right)^{1/3}} \right), -\frac{1}{6} \left(1259 + 3\sqrt{206634}\right)^{1/3} + \frac{65}{6\left(1259 + 3\sqrt{206634}\right)^{1/3}} + \frac{2}{3} + \frac{1}{2} \mathrm{I}\sqrt{3} \left(\frac{1}{3} \left(1259 + 3\sqrt{206634}\right)^{1/3} + \frac{65}{3\left(1259 + 3\sqrt{206634}\right)^{1/3}} \right) \right] \quad \textbf{(10)}$$

> *reslist2* := [*res*];

$$reslist2 := \left[\frac{1}{3} \left(1259 + 3\sqrt{206634}\right)^{1/3} - \frac{65}{3\left(1259 + 3\sqrt{206634}\right)^{1/3}} + \frac{2}{3}, -\frac{1}{6} \left(1259 + 3\sqrt{206634}\right)^{1/3} + \frac{65}{6\left(1259 + 3\sqrt{206634}\right)^{1/3}} + \frac{2}{3} + \frac{1}{2} \mathrm{I}\sqrt{3} \left(\frac{1}{3} \left(1259 + 3\sqrt{206634}\right)^{1/3} + \frac{65}{3\left(1259 + 3\sqrt{206634}\right)^{1/3}} \right), -\frac{1}{6} \left(1259 + 3\sqrt{206634}\right)^{1/3} + \frac{65}{6\left(1259 + 3\sqrt{206634}\right)^{1/3}} + \frac{2}{3} - \frac{1}{2} \mathrm{I}\sqrt{3} \left(\frac{1}{3} \left(1259 + 3\sqrt{206634}\right)^{1/3} + \frac{65}{3\left(1259 + 3\sqrt{206634}\right)^{1/3}} \right) \right] \quad \textbf{(11)}$$

> *evalf*(*reslist*[1]);

$$3.692418863 \quad \textbf{(12)}$$

>

> *seq*(*solve*({*x* = *reslist*[*i*], *eq2*}, {*x*, *y*}), *i* = 1 ..3);

$$\left\{ y = -\frac{1}{24} \frac{4\left(1259 + 3\sqrt{206634}\right)^{2/3} - 260 - 21\left(1259 + 3\sqrt{206634}\right)^{1/3}}{\left(1259 + 3\sqrt{206634}\right)^{1/3}}, x = \frac{1}{3} \frac{\left(1259 + 3\sqrt{206634}\right)^{2/3} - 65 + 2\left(1259 + 3\sqrt{206634}\right)^{1/3}}{\left(1259 + 3\sqrt{206634}\right)^{1/3}} \right\}, \left\{ x = -\frac{1}{6} \frac{1}{\left(1259 + 3\sqrt{206634}\right)^{1/3}} \left(\left(1259 + 3\sqrt{206634}\right)^{2/3} - 65 - 4\left(1259 + 3\sqrt{206634}\right)^{1/3} + \mathrm{I}\sqrt{3}\left(1259 + 3\sqrt{206634}\right)^{2/3} + 65\,\mathrm{I}\sqrt{3} \right), y = \frac{1}{24} \frac{1}{\left(1259 + 3\sqrt{206634}\right)^{1/3}} \left(2\left(1259 + 3\sqrt{206634}\right)^{2/3} - 130 + 21 \left(1259 \right.\right. \quad \textbf{(13)}$$

$$+ 3\sqrt{206634}\Big)^{1/3} + 2\,\mathrm{I}\sqrt{3}\,\Big(1259 + 3\sqrt{206634}\Big)^{2/3} + 130\,\mathrm{I}\sqrt{3}\Big)\Big\}, \Big\{x$$

$$= \frac{1}{6}\,\frac{1}{\left(1259 + 3\sqrt{206634}\right)^{1/3}}\Big(-\Big(1259 + 3\sqrt{206634}\Big)^{2/3} + 65 + 4\,\Big(1259$$

$$+ 3\sqrt{206634}\Big)^{1/3} + \mathrm{I}\sqrt{3}\,\Big(1259 + 3\sqrt{206634}\Big)^{2/3} + 65\,\mathrm{I}\sqrt{3}\Big), y =$$

$$-\frac{1}{24}\,\frac{1}{\left(1259 + 3\sqrt{206634}\right)^{1/3}}\Big(-2\,\Big(1259 + 3\sqrt{206634}\Big)^{2/3} + 130 - 21\,\Big(1259$$

$$+ 3\sqrt{206634}\Big)^{1/3} + 2\,\mathrm{I}\sqrt{3}\,\Big(1259 + 3\sqrt{206634}\Big)^{2/3} + 130\,\mathrm{I}\sqrt{3}\Big)\Big\}$$

> *seq*(*fsolve*({*x* = *reslist*[*i*], *eq2*}, {*x*, *y*}), *i* = 1 ..3);

{y = -0.6378760982, x = 3.692418863}, {x = -0.8462094323 − 5.341633545 I, y = 1.631438050 + 2.670816772 I}, {x = -0.8462094323 + 5.341633545 I, y = 1.631438050 − 2.670816772 I} **(14)**

> *fsolve*(*x* = *Pi* ·*I*, *x*);

3.141592654 I **(15)**

Another example:

> $\frac{\text{factorial}(10000)}{\text{factorial}(9999)}$;

10000 **(16)**

> *solve*(*x*·factorial(9999) = 10000, *x*);

1/ **(17)**

2846259680917054518906413212119868890148051401702799230794179994274411340\
0376444377299078675778477 5[...35452 digits...]
000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000

>

Sequences, Limits and Series

Little dictionary:
limit : Grenzwert
sequence : Folge
series : Reihe

Definition (*sequence*): A **sequence** of real numbers is a mapping from ℕ → ℝ.
Example: Let a_n:= 1/n, n ⩾1. This gives the sequence (1, 1/2, 1/3, ...)

Definition (*convergence,limit*): Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers. A sequence is called convergent towards a $\in \mathbb{R}$, if and only if:

For all $\epsilon > 0$ it exists an $N(\epsilon) \in \mathbb{N}$ such that

$|a_n - a| < \epsilon$ for all $n \geq N(\epsilon)$.

We write $\lim_{n \to \infty} a_n = a$.

Definition (*series*): Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers. The sequence $s_n := \sum_{k=0}^{n} a_k,\ n \in \mathbb{N}$ of sums is called **series**, and is described with the help of $\sum_{n=0}^{\infty} a_n$.

Definition (*absolute convergence*): A series $\sum_{n=0}^{\infty} a_n$ is said to **converge absolutely** if the series $\sum_{n=0}^{\infty} |a_n|$ converges, where $|a_n|$ denotes the absolute value of a_n.

Definition (*limits at functions*): Let $f : D \to \mathbb{R}$ a real valued function on the domain $D \subseteq \mathbb{R}$ with a point a $\in \mathbb{R}$, such that there exists at least one sequence $(a_n)_{n \in \mathbb{N}},\ a_n \in D$ with $\lim_{n \to \infty} a_n = a$.

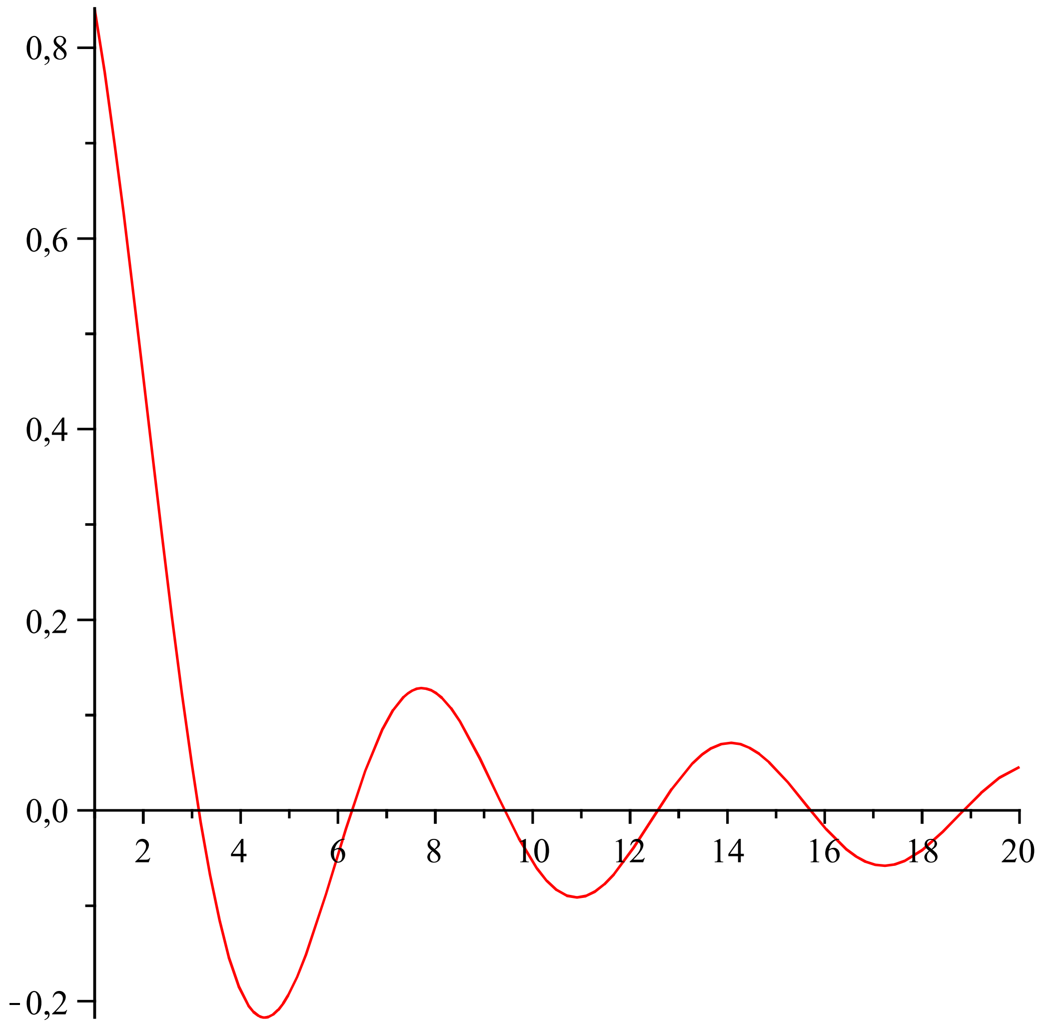
We write

$$\lim_{x \to a} f(x) = c$$

if and only if it is valid:

$\lim_{n \to \infty} f(x_n) = c$ for all $(x_n)_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} x_n = a$.

```
> restart;
> plot(1/x · sin(x), x = 1..20);
```



>

> ?*plot*

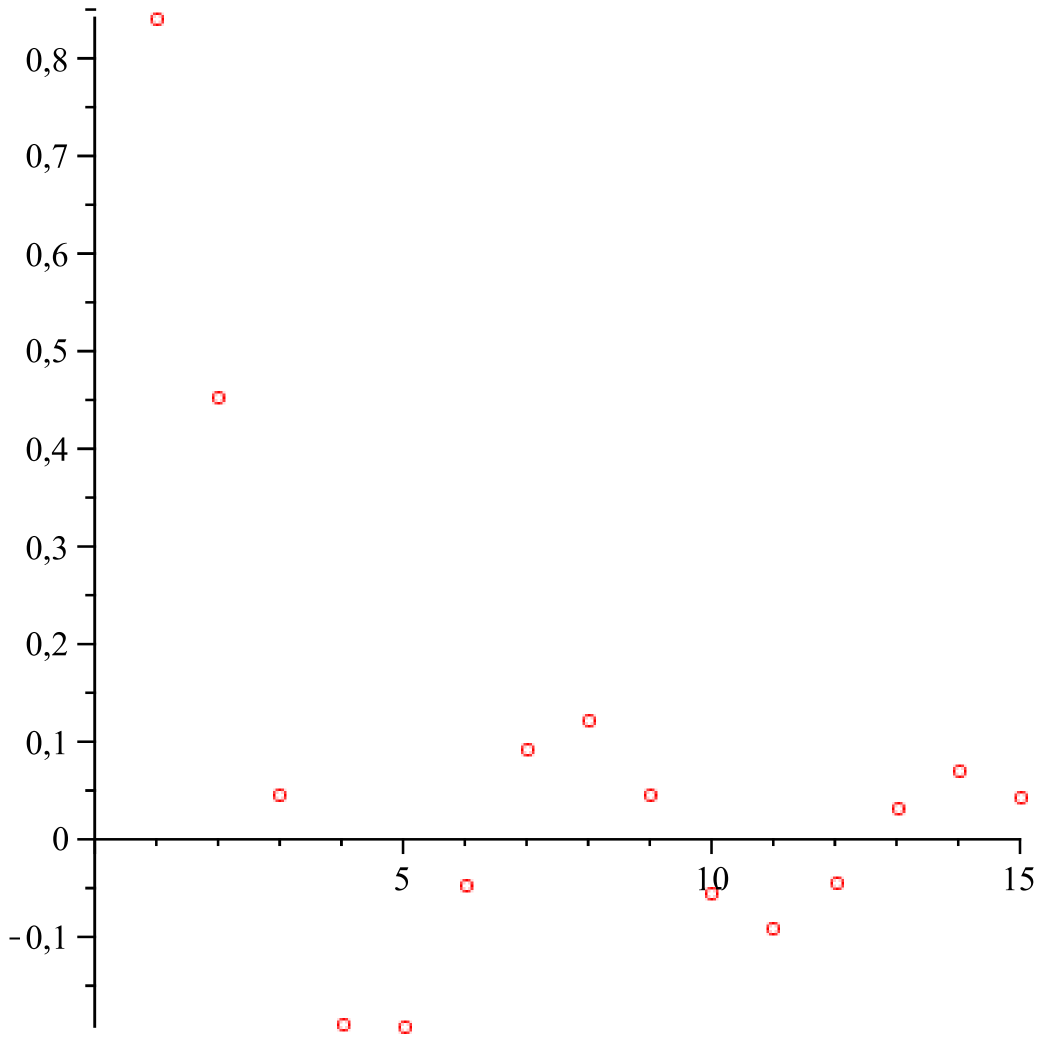
> ?\`$\`

> $l1 := \left[\left[n, \frac{1}{n} \cdot \sin(n) \right] \$ n = 1 ..20 \right]:$

$l2 := \left[seq\left(\left[n, \frac{1}{n} \cdot \sin(n) \right], n = 1 ..20 \right) \right];$

$plot(l2, x = 0 ..15, style = point, symbol = circle);$

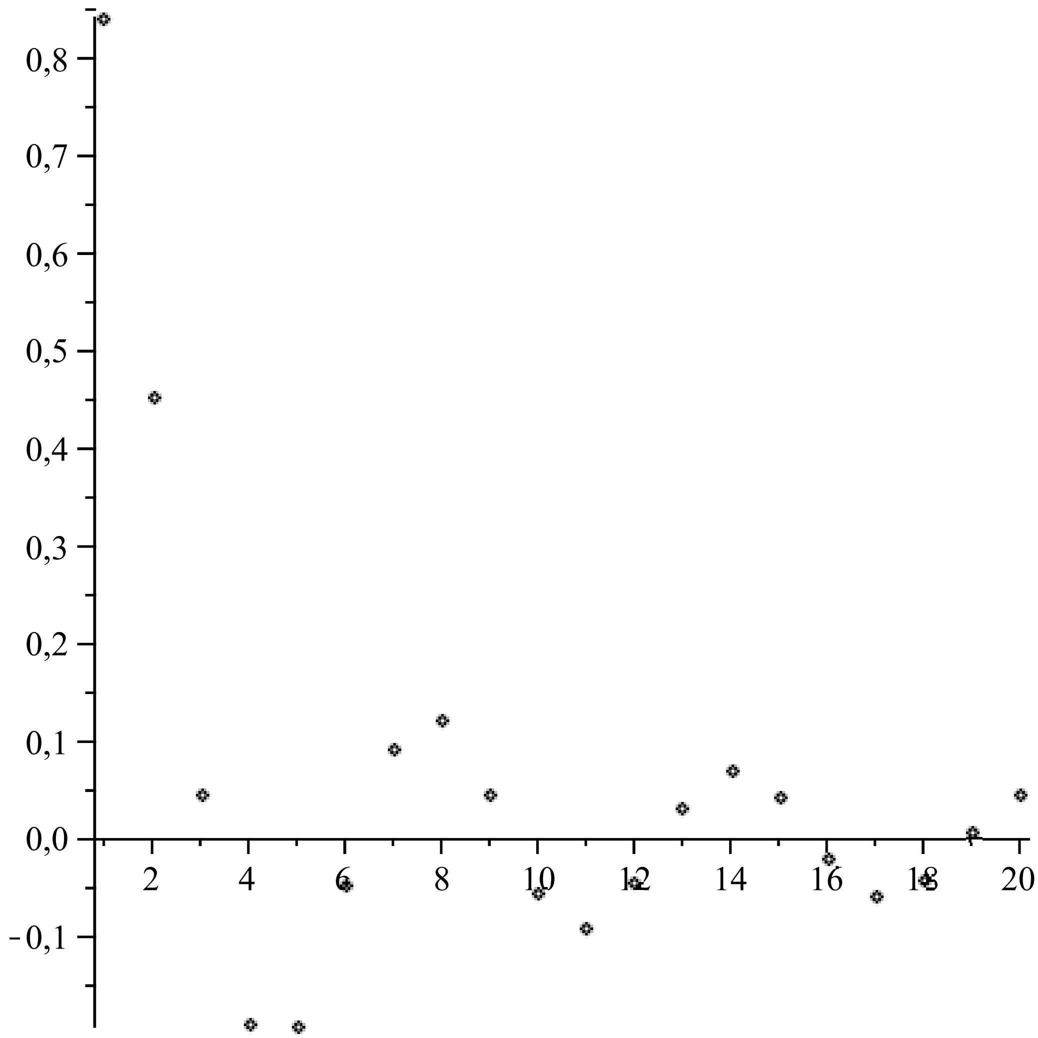
$$l2 := \left[[1, \sin(1)], \left[2, \frac{1}{2} \sin(2)\right], \left[3, \frac{1}{3} \sin(3)\right], \left[4, \frac{1}{4} \sin(4)\right], \left[5, \frac{1}{5} \sin(5)\right], \left[6, \frac{1}{6} \sin(6)\right], \left[7, \frac{1}{7} \sin(7)\right], \left[8, \frac{1}{8} \sin(8)\right], \left[9, \frac{1}{9} \sin(9)\right], \left[10, \frac{1}{10} \sin(10)\right], \left[11, \frac{1}{11} \sin(11)\right], \left[12, \frac{1}{12} \sin(12)\right], \left[13, \frac{1}{13} \sin(13)\right], \left[14, \frac{1}{14} \sin(14)\right], \left[15, \frac{1}{15} \sin(15)\right], \left[16, \frac{1}{16} \sin(16)\right], \left[17, \frac{1}{17} \sin(17)\right], \left[18, \frac{1}{18} \sin(18)\right], \left[19, \frac{1}{19} \sin(19)\right], \left[20, \frac{1}{20} \sin(20)\right] \right]$$



>

> ?*list*

> $plots[pointplot]\left(\left\{seq\left(\left[x, \frac{1}{x}\cdot\sin(x)\right], x=1..20\right)\right\}\right);$



Computations of limits:

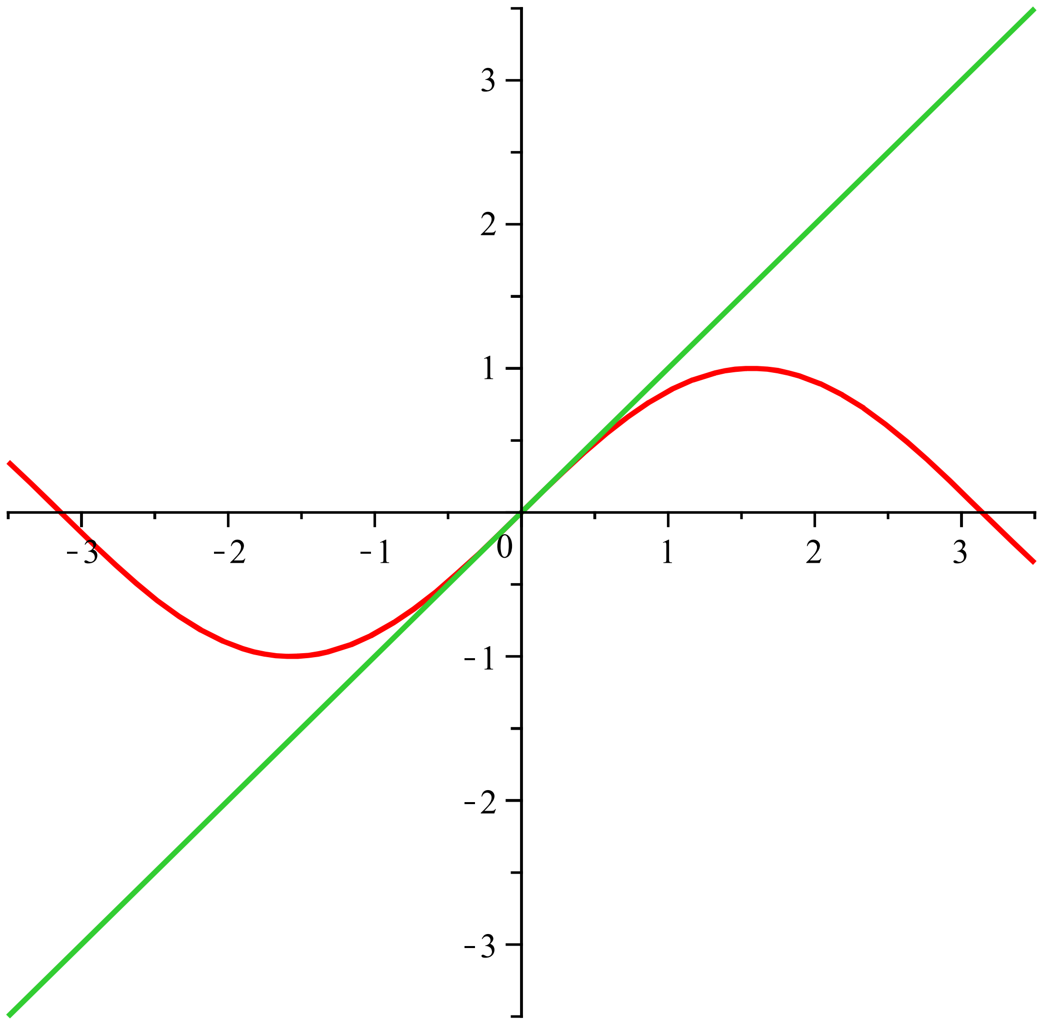
> $limit(\sin(x), x=0);$

$$0 \quad \textbf{(18)}$$

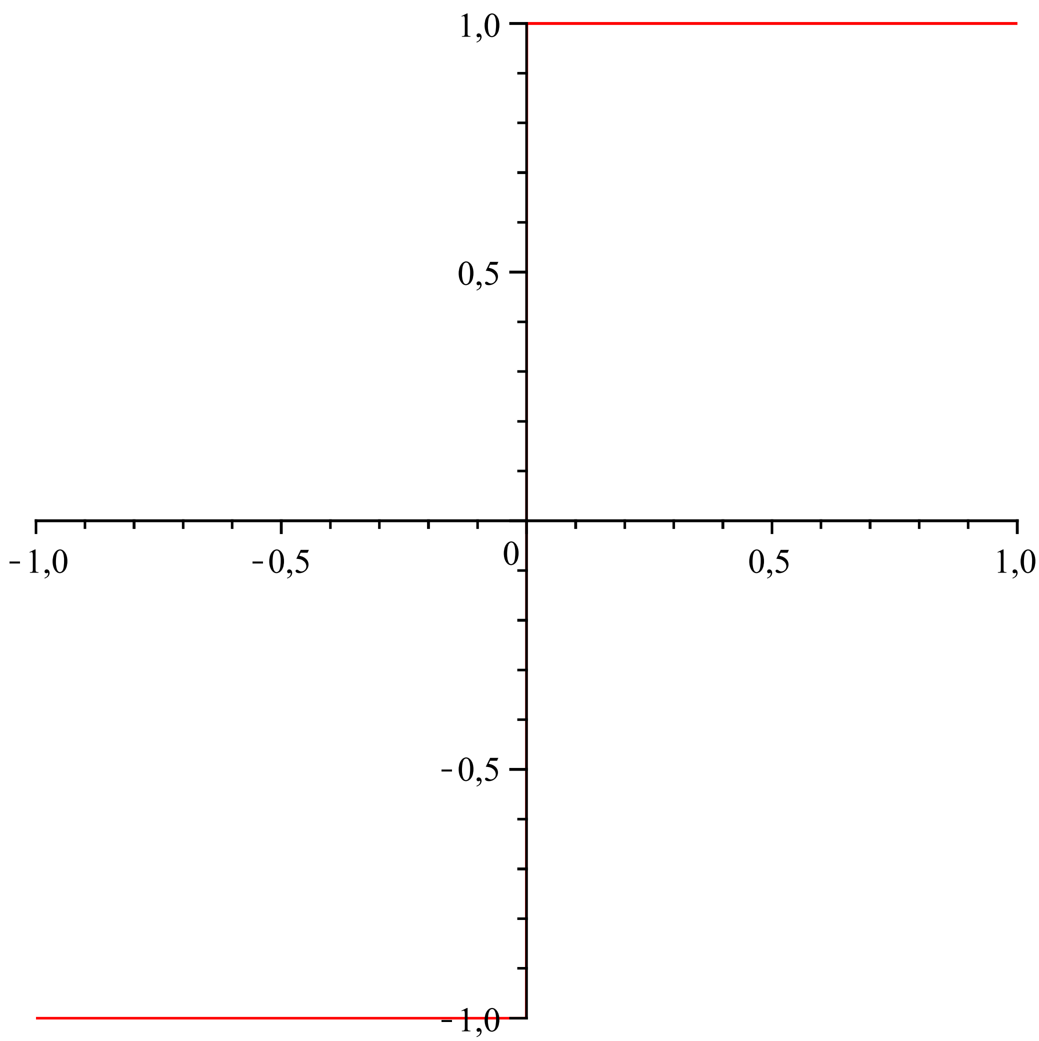
> $limit\left(\frac{\sin(x)}{x}, x=0\right);$

$$1 \quad \textbf{(19)}$$

> $plot([\sin(x), x], x=-3.5..3.5, thickness=2);$



> $plot(\mathrm{signum}(x), x=-1..1);$



> $limit(\text{signum}(x), x=0);$

$$undefined \quad (20)$$

> $limit(\text{signum}(x), x=0, left);$

$$-1 \quad (21)$$

> $limit(\text{signum}(x), x=0, right);$

$$1 \quad (22)$$

> $limit(\exp(x), x=infinity);$

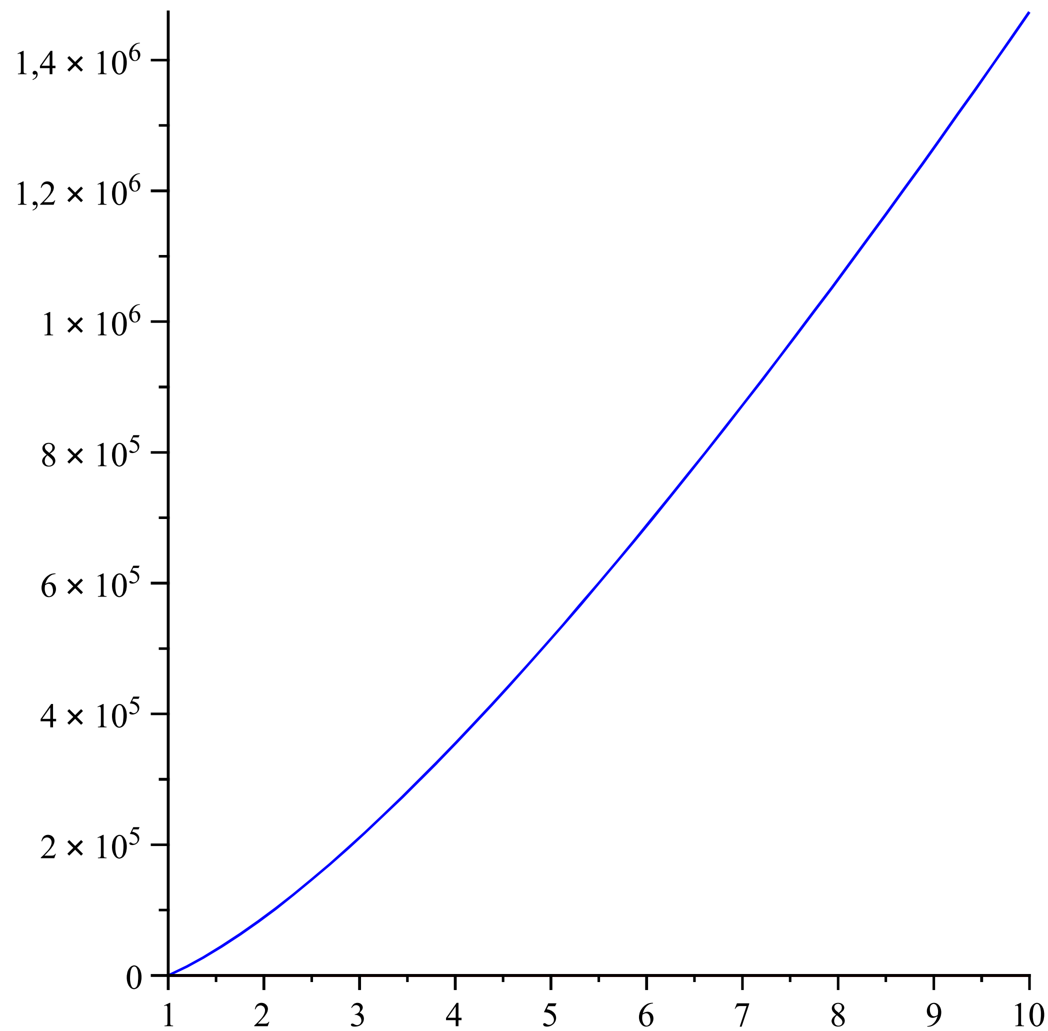
$$\infty \quad (23)$$

Growth of function

Definition: *Complexity of an algorithm*
Let A be a deterministic algorithm that has finite running time for all possible inputs.
The runtime (time complexity) of A is a function f : $\mathbb{N} \rightarrow \mathbb{N}$,
with: f(n) is the maximum number of steps of A when given an input of length n.

Assume two algorithms A1 and A2, e.g. for the problem of sorting n numbers.
Let A1 use n^2 - 20n + 1 steps in order to sort a sequence of n numbers (in worst case).
Let A2 use 200n * 320 log(n) many steps for the same purpose. Which one grows faster?

> $plot([n^2 - 20\cdot n + 1, 200\cdot n\cdot 320\cdot \log(n)], n = 1..10, color = [red, blue]);$



> *restart*;

Asymptotic growth of function and O-notation.

Definition 1 : $\exists\ k>0, n_0>0\ \forall n > n_0 : f(n) \le k\cdot g(n)$

Intuition 1 : asymptotically, f is bounded above by g; "asymptotically, f does not grow faster than g"

Notation 1 : f(n) ∈O(g(n)) (i.e. $f \in \{h: \mathbb{N} \to \mathbb{N} \mid \exists\ k>0, n_0>0\ \forall n > n_0 : h(n) \le k\cdot g(n)\}$)

Connection to limit 1 : **if** $\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$ **then** $f(n) \in O(g(n))$

Proof 1 : Assume $\lim_{n\to\infty} \frac{f(n)}{g(n)} = K < \infty$.

By definition of limit we have $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall\ n > n_0 : \left|\frac{f(n)}{g(n)} - K\right| < \varepsilon$.

and thus $-\varepsilon <$ f(n)/g(n)-K $< \varepsilon$ (more exactly: $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall\ n > n_0 : -\varepsilon < f(n)/g(n)-K < \varepsilon$)

hence $-\varepsilon + K < f(n)/g(n) < \varepsilon + K$
therefore $f(n) < (\varepsilon + K)\ g(n)$

and with k := ε + K: ∃ k>0, n_0>0 ∀n > n_0 : $f(n) \le k \cdot g(n)$

Definition 2 : ∃ k>0, n_0>0 ∀n > n_0 : $f(n) < k \cdot g(n)$

Intuition 2 : "asymptotically, f grows slower than g"

Notation 2 : f(n) ∈o(g(n))   (i.e. f ∈{h: ℕ→ℕ | ∃ k>0, n_0>0 ∀n > n_0 : $h(n) < k \cdot g(n)$})

Connection to limit 2 : **if** $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ **then** $f(n) \in o(g(n))$

if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$ **then** $g(n) \in o(f(n))$

Now, we want to compare two algorithms A1 and A2 concerning their time complexity.

> $tcA1 := \frac{1}{2} \cdot n^2 \cdot \log(n);\ tcA2 := n^2;\ limit\left(\frac{tcA1}{tcA2}, n = infinity\right);$

$$tcA1 := \frac{1}{2}\, n^2 \ln(n)$$

$$tcA2 := n^2$$

$$\infty \qquad \textbf{(24)}$$

> $tcA1 := \sum_{i=1}^{n} i;\ txA2 := n^2;\ limit\left(\frac{tcA1}{tcA2}, n = infinity\right);\ limit\left(\frac{tcA2}{tcA1}, n = infinity\right);$

$$tcA1 := \frac{1}{2}\,(n+1)^2 - \frac{1}{2}\,n - \frac{1}{2}$$

$$txA2 := n^2$$

$$\frac{1}{2}$$

$$2 \qquad \textbf{(25)}$$

Further examples.

> $limit\left(\frac{n^2}{n^3+1}, n = \infty\right);$

$$0 \qquad \textbf{(26)}$$

> $limit\left(\frac{\mathrm{Pi} \cdot n^3 + 17 \cdot n + n}{n^3 + 39}, n = \infty\right);$ # *wrong space!*

$$limit\left(\frac{\pi\, n^3 + 18\, n}{n^3 + 39}, n = \infty\right) \qquad \textbf{(27)}$$

> $limit\left(\frac{n^k}{n!}, n = infinity\right);$

$$0 \qquad \textbf{(28)}$$

> $limit\left(\frac{n^n}{n!}, n = infinity\right);$

$$\infty \qquad \textbf{(29)}$$

> $limit\left(\frac{n^k}{n!}, n=0\right);$

$$\lim_{n\to 0}\left(\frac{n^k}{n!}\right) \qquad \textbf{(30)}$$

> $limit\left(\frac{n^k}{n!}, n=0\right)$ assuming $k > 0;$

$$0 \qquad \textbf{(31)}$$

> $limit\left(\frac{n^k}{n!}, n=0\right)$ assuming $k < 0;$

$$\infty \qquad \textbf{(32)}$$

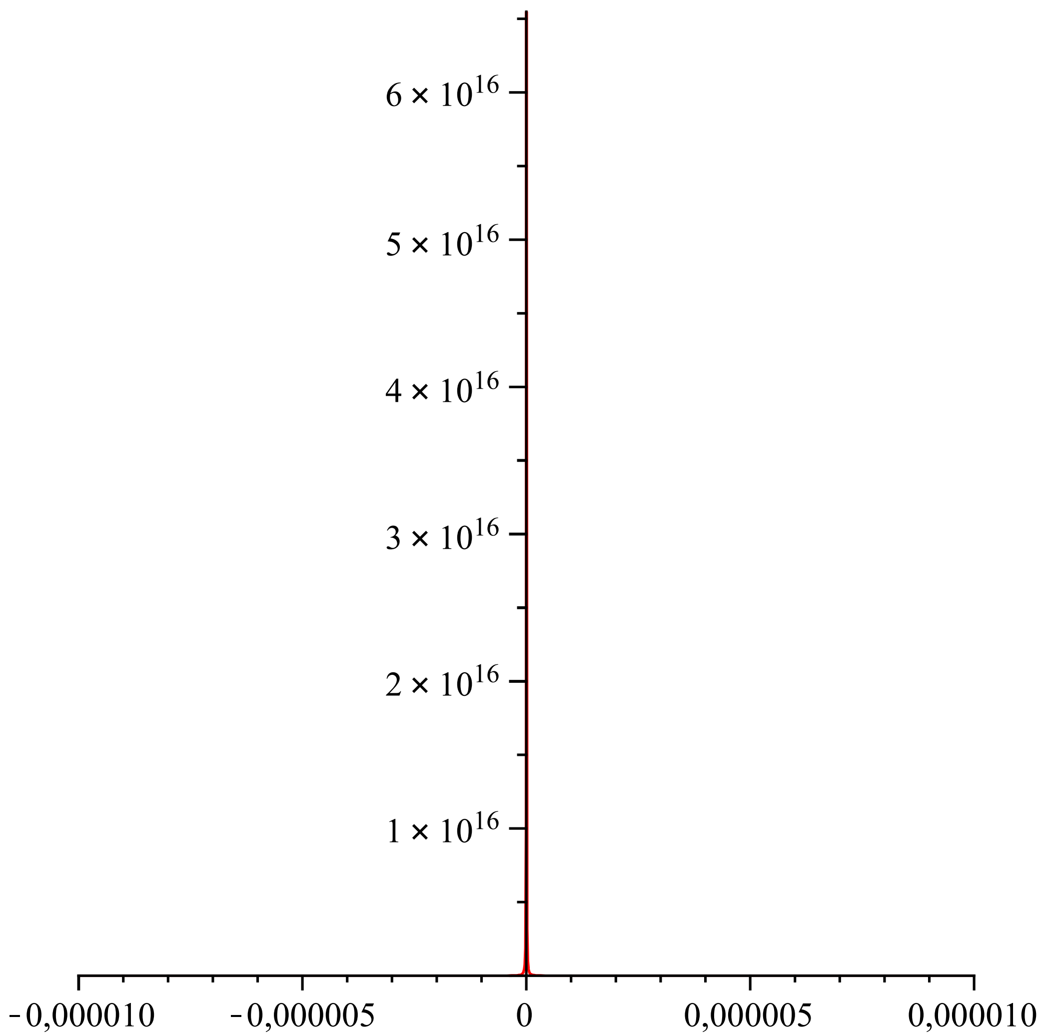
> ?factorial

> $evalb(99! = \mathrm{GAMMA}(100));$

$$true \qquad \textbf{(33)}$$

> $simplify(n! - \mathrm{GAMMA}(n+1));$

> $plot\left(\frac{n^{-2}}{n!}, n=-0.00001..0.00001\right); evalf\left(\frac{0.00001^{-2}}{0.00001!}\right);$



$$1.000005772\ 10^{10} \qquad \textbf{(34)}$$

> $limit\left(\frac{n^k}{n!}, n=0\right)$ assuming $k=0$;

$$\lim_{n\to 0}\left(\frac{n^k}{n!}\right) \qquad (35)$$

> $eval\left(limit\left(\frac{n^k}{n!}, n=0\right)\right)$ assuming $k=0$;

$$\lim_{n\to 0}\left(\frac{n^k}{n!}\right) \qquad (36)$$

> $limit\left(\frac{n^0}{n!}, n=0\right)$;

$$1 \qquad (37)$$

>

Series

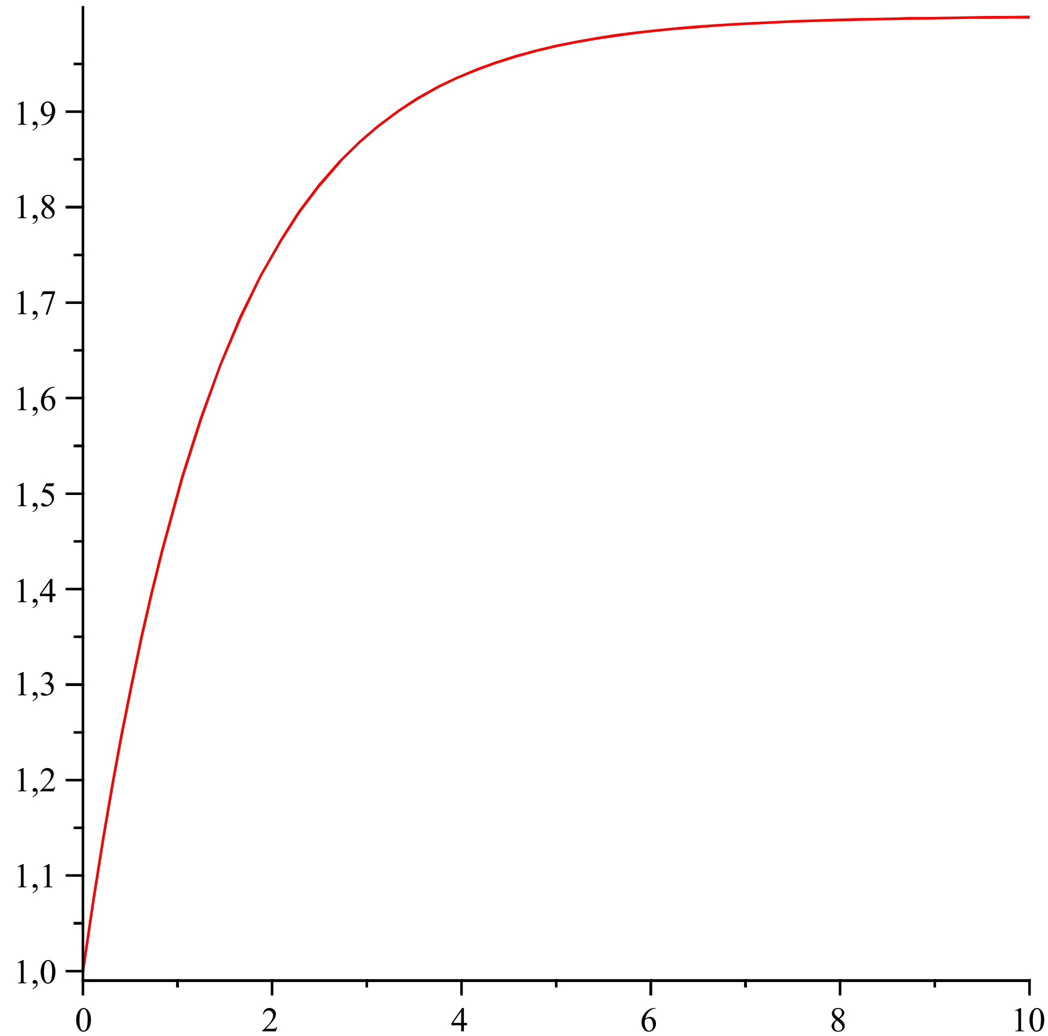
First idea

> $restart$;

> $sum(a[k], k=0..infinity)$;

$$\sum_{k=0}^{\infty} a_k \qquad (38)$$

> $plot\left(\sum_{i=0}^{n}\left(\frac{1}{2}\right)^i, n=0..10\right)$;



> $sum\left(\left(\frac{1}{2}\right)^n, n = 0 ..infinity\right);$

$$2 \tag{39}$$

> $limit\left(\sum_{i=0}^{n}\left(\frac{1}{2}\right)^i, n = infinity\right);$

$$2 \tag{40}$$

> $\sum_{i=0}^{n}\left(\frac{1}{2}\right)^i;$

$$-2\left(\frac{1}{2}\right)^{n+1} + 2 \tag{41}$$

> $f := \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} - \sum_{i=0}^{n}\left(\frac{1}{2}\right)^i;$

$$f := 0 \tag{42}$$

> $\sum_{i=0}^{\infty} x^i;$

$$-\frac{1}{x-1} \tag{43}$$

> *#computing with series*

> $\sum_{i=0}^{12} x^i + \sum_{i=15}^{\infty} x^i;$

$$1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}+x^{12}-\frac{x^{15}}{x-1} \tag{44}$$

> $simplify(\%);$

$$-\frac{1-x^{13}+x^{15}}{x-1} \tag{45}$$

Harmonic Series

> $Harmonic := \sum_{i=1}^{\infty} \frac{1}{i};$

$$Harmonic := \infty \tag{46}$$

> $\sum_{i=1}^{\infty} (-1)^i \frac{1}{i};$

$$-\ln(2) \tag{47}$$

> $AlternatingHarmonic := n \rightarrow \frac{(-1)^n}{n};$

$$AlternatingHarmonic := n \rightarrow \frac{(-1)^n}{n} \tag{48}$$

> $map(AlternatingHarmonic, [seq(1..10)]);$

$$\left[-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, -\frac{1}{7}, \frac{1}{8}, -\frac{1}{9}, \frac{1}{10}\right] \tag{49}$$

> $sum(AlternatingHarmonic(n), n = 1..infinity)$

$$-\ln(2) \tag{50}$$

The Riemann Series Theorem

Theorem: Let f $: \mathbb{N} \rightarrow \mathbb{R}$ be a sequence such that the series $\sum_{k=1}^{\infty} f(k)$ converges but not absolutely.

Then: For each real x there is a bijection (a re-ordering) $\beta:\mathbb{N} \rightarrow \mathbb{N}$ such that $\sum_{k=1}^{\infty} f(\beta(k)) = x.$

We want to construct such a reordering for given f and x. First we need two short functions which will be helpful.

```
> restart;
> AlternatingHarmonic := n -> (-1)^n/n;
```

$$AlternatingHarmonic := n \rightarrow \frac{(-1)^n}{n} \quad \textbf{(51)}$$

```
>
> FindNextPositiveStartingIndex := proc(f, k)
    local i;
    i := k;
    while f(i) <= 0 do
     i := i + 1
    end do;
    return i
  end proc;
```

FindNextPositiveStartingIndex := **proc**(*f*, *k*) **(52)**

local *i*; *i* := *k*; **while** *f*(*i*) <= 0 **do** *i* := *i* + 1 **end do**; **return** *i*

end proc

```
> FindNextNegativeStartingIndex :=
  proc(f, k)
     local i;
     i := k;
     while 0 ≤ f(i) do
        i := i + 1
     end do;
     return i;
  end proc
```

FindNextNegativeStartingIndex := **proc**(*f*, *k*) **(53)**

local *i*; *i* := *k*; **while** 0 <= *f*(*i*) **do** *i* := *i* + 1 **end do**; **return** *i*

end proc

```
> AlternatingHarmonic(2);
```

$$\frac{1}{2} \quad \textbf{(54)}$$

```
> FindNextPositiveStartingAt(AlternatingHarmonic, 1);
```

FindNextPositiveStartingAt(AlternatingHarmonic, 1) **(55)**

```
>
>
> Riemann := proc(f, x, k)    # here: only vor strictly alternating series
      local s, pix, nix, j, p, n, ret;
      s := 0;
      ret := -1;
      pix := FindNextPositiveStartingIndex(f, 1);
      nix := FindNextNegativeStartingIndex(f, 1);
     for j from 1 to k do
        if evalf(s) < evalf(x) then
```

```
            s := s + f(pix);
            ret := f(pix);
            pix := pix + 2;
         else
            s := s + f(nix);
            ret := f(nix);
            nix := nix + 2;
         end if;
      end do;
      return [ret, evalf(s)]
   end proc
```

```
Riemann := proc(f, x, k)                                                    (56)
   local s, pix, nix, j, p, n, ret;
   s := 0;
   ret := -1;
   pix := FindNextPositiveStartingIndex(f, 1);
   nix := FindNextNegativeStartingIndex(f, 1);
   for j to k do
      if evalf(s) < evalf(x) then
         s := s + f(pix); ret := f(pix); pix := pix + 2
      else
         s := s + f(nix); ret := f(nix); nix := nix + 2
      end if
   end do;
   return [ret, evalf(s)]
end proc
```

> seq(Riemann(AlternatingHarmonic, Pi, i)[1], i = 1000 ..1020);

$$\frac{1}{1998}, \frac{1}{2000}, \frac{1}{2002}, \frac{1}{2004}, \frac{1}{2006}, \frac{1}{2008}, \frac{1}{2010}, \frac{1}{2012}, \frac{1}{2014}, \frac{1}{2016}, \frac{1}{2018}, \frac{1}{2020}, \frac{1}{2022}, \frac{1}{2024}, \frac{1}{2026}, \frac{1}{2028}, \frac{1}{2030}, \frac{1}{2032}, \frac{1}{2034}, \frac{1}{2036}, \frac{1}{2038} \quad (57)$$

> $seq\left(Riemann\left(AlternatingHarmonic, \frac{100}{1001}, i\right)[2], i = 1000 ..1020\right);$

0.09996687477, 0.09703432345, 0.09763600817, 0.09823696971, 0.09883720981, 0.09943673019, 0.1000355326, 0.09712008069, 0.09771816681, 0.09831553838, 0.09891219709, 0.09950814465, 0.1001033827, 0.09720483202, 0.09779936234, 0.09839318657, 0.09898630638, 0.09957872344, 0.1001704394, 0.09728859504, 0.09787961158 **(58)**

> evalb(0 < evalf(sqrt(2)));

true **(59)**

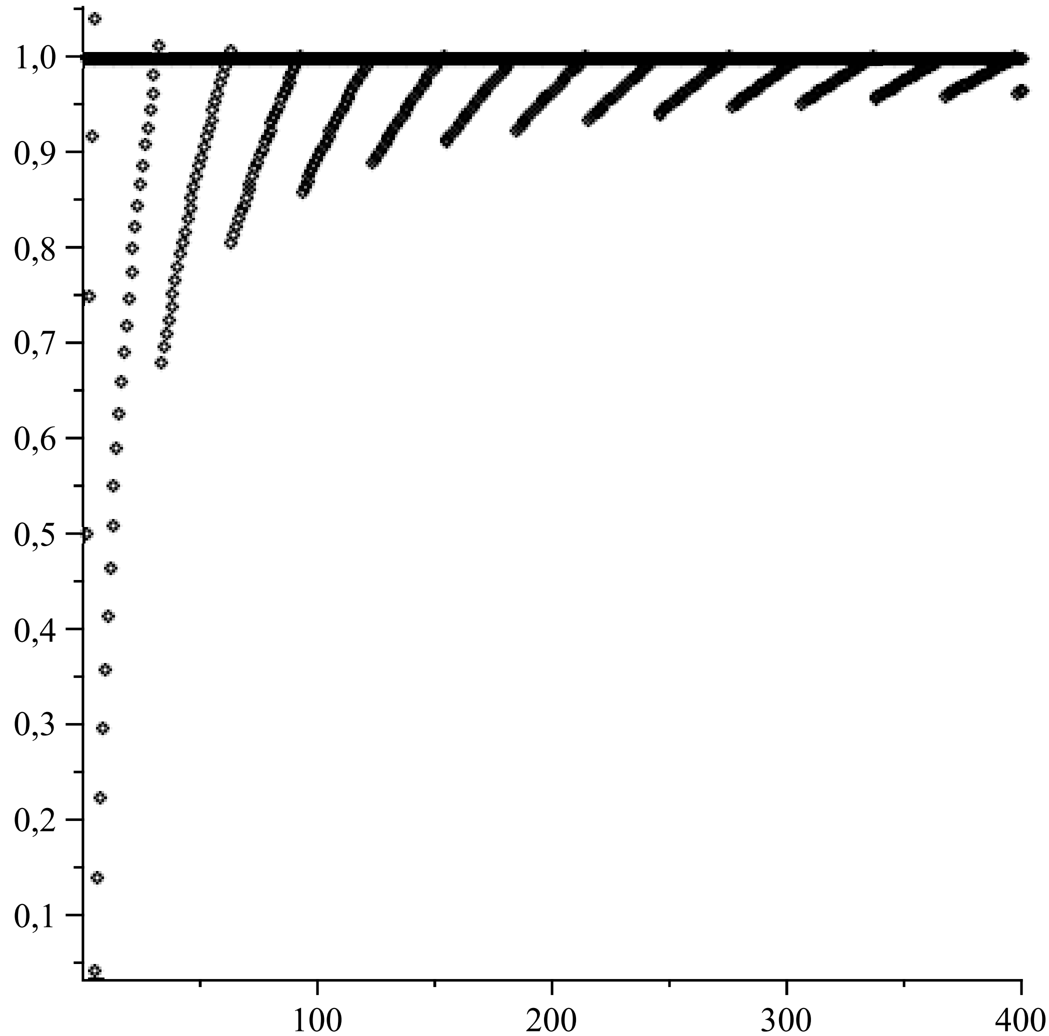
> sum(AlternatingHarmonic(n), n = 1 ..∞);

$-\ln(2)$ **(60)**

> $rseq := seq\left(\left[i, Riemann\left(AlternatingHarmonic, \frac{1000}{1001}, i\right)[2]\right], i = 1 ..400\right):$

>

> $plots[pointplot]\left(\left\{seq\left(\left[x, \frac{1000}{1001}\right], x = 1..400\right), \text{rseq}\right\}\right);$



>

>

>

$$0 \quad \textbf{(61)}$$

>