Background know-how

Limits of Floating-Point arithmetic in C

```
#include <stdio.h>
int main(void) {
    double x=0.7;
    int i = 0;
    while(i < 10) {
        x = 11.0 * x – 7.0;
        printf("%d: %.20lf\n",i,x);
        i=i+1;
    }
}
```

The result of the C-program is rubbish. In the last round it is
y = -1127140547773912.5

Limits of Floating-Point arithmetic in Maple

> $restart; x := \frac{7.0}{10};$

$$x := 0.7000000000 \quad (1)$$

> **for** i **from** 1 **to** 30 **do**
  $x := 11 \cdot x - 7;$
 end do:

> $x;$

$$0.700000000 \quad (2)$$

> $restart; x := \frac{1.0}{3}:$

> **for** i **from** 1 **to** 30 **do**
  $x := 3 \cdot x - \frac{2}{3};$
 end do:

> $x;$

$$-10294.22328 \quad (3)$$

>

> $x := 0 : t := time(\,) :$
for i **from** 1 **to** 5000000 **do**
  $r := rand(\,)$ **mod** 10;
  for j **from** 1 **to** r **do**
   $x := x + 1;$
  end do:
 end do:
$x, time(\,) - t;$

$$22492822, 27.480 \quad (4)$$

Numbers, their representations and more and less native number representations for a digital computer

numbers can be elements from various sets. e.g. $x \in \mathbb{Z}$, $x \in \mathbb{N}$.
each number has various representations. e.g.
17
XVII
IIIII IIIII IIIII II

usually, we encode numbers with the help of base-10 digits, i.e. the alphabet
$\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
A string $s = (a_{n-1} a_{n-1} \dots a_1 a_0) \in \Sigma_n$ is then interpreted as

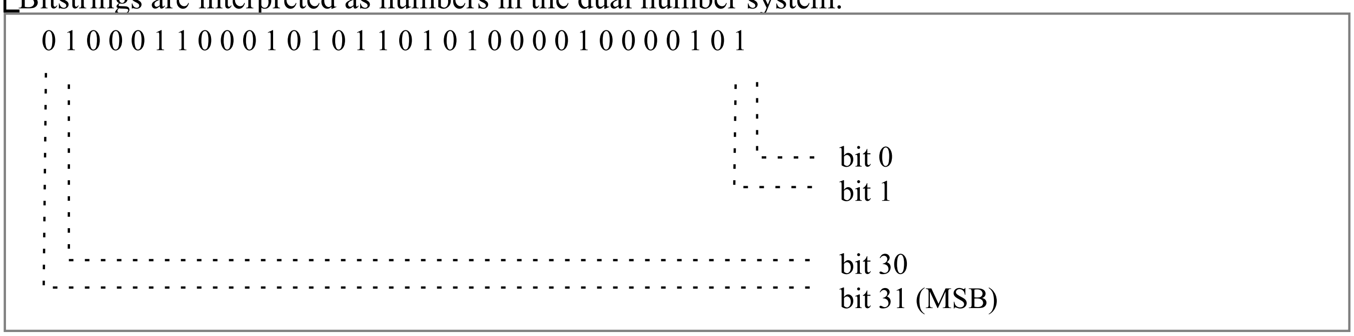
$$\sum_{i=0}^{n-1} a_i \cdot 10^i \ : \ \text{Example: } 17 = 1 \cdot 10^1 + 7 \cdot 10^0$$

What happens, if we use another base, another alphabet?
E.g. with "bits", we have:
$\Sigma_2 = \{0, 1\}$ $\quad 17_{10} = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 10001_2$

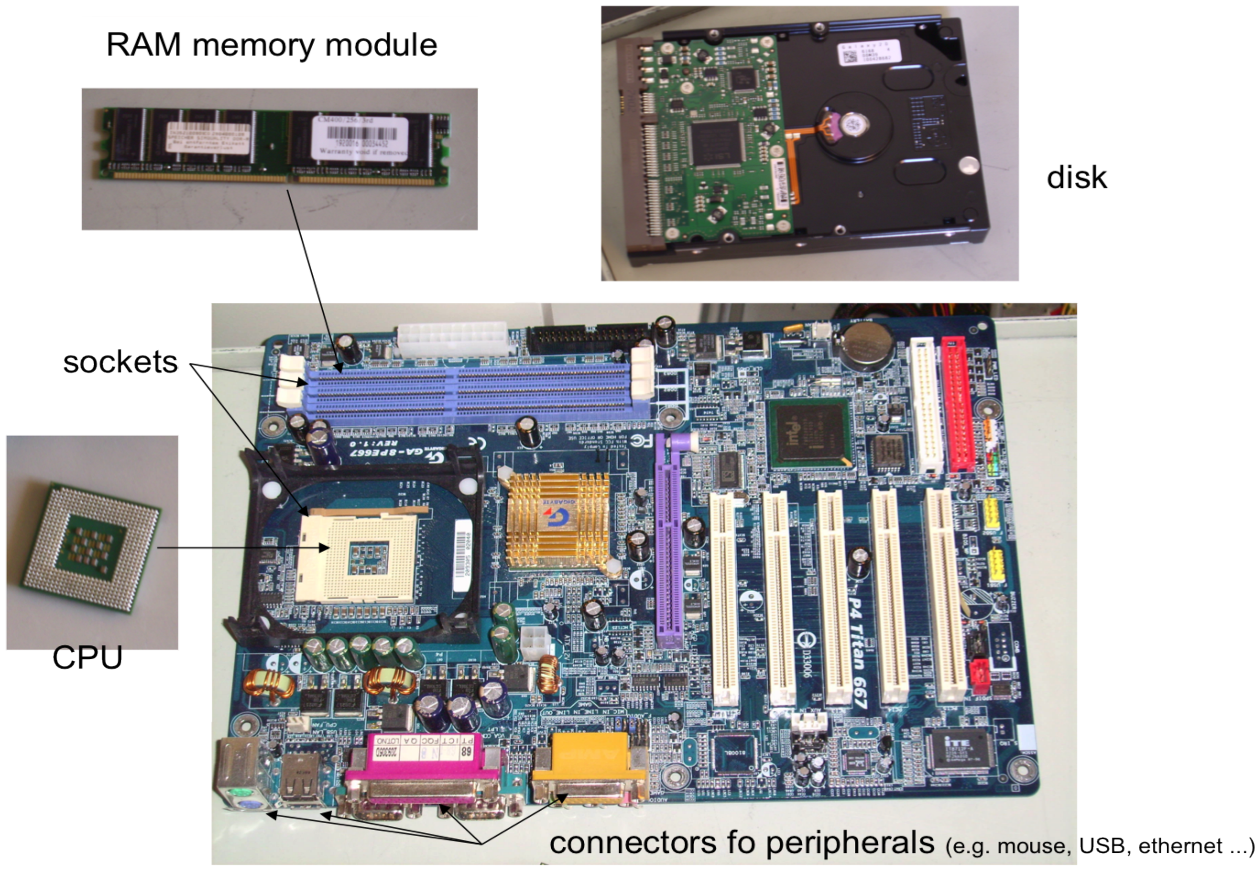
$\Sigma_{16} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f\}$
$17_{10} = 1 \cdot 16^1 + 1 \cdot 16^0 = 0x11$ (so called hex numbers)

integer variables of fixed length are the most natural and mostly used kind of variables

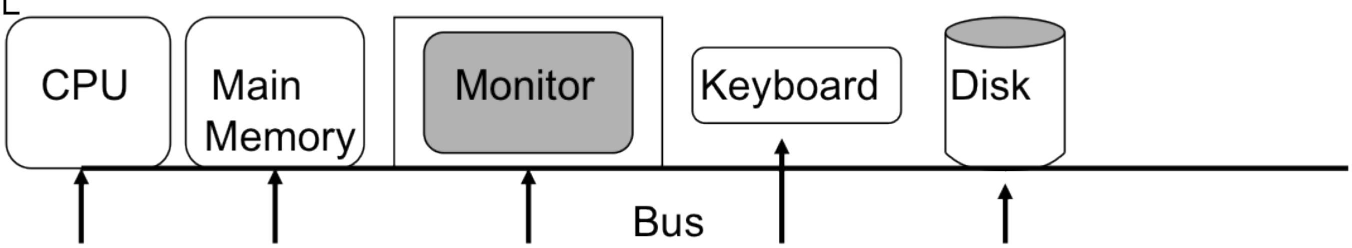
Bitstrings are interpreted as numbers in the dual number system.



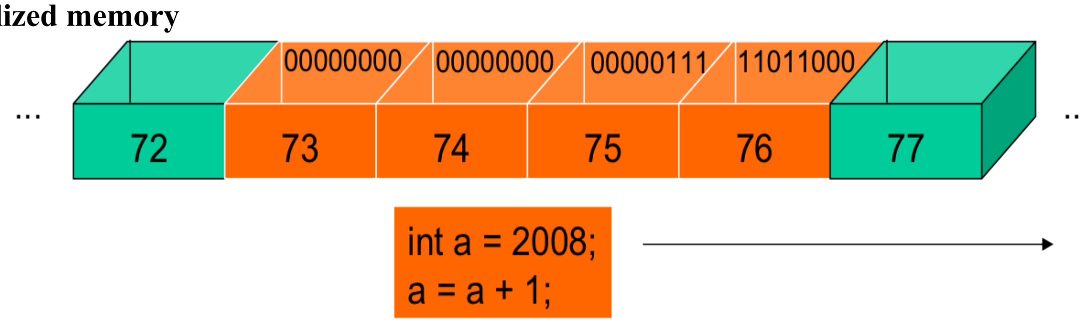
The value then is $bit_{31} \cdot 2^{31} + bit_{30} \cdot 2^{30} + \dots + bit_0 \cdot 2^0$.



My idealized computer



My idealized memory



How to compute with binary numbers?

□ base-2 base-10

sum:

```
    1 0 1 1            9 9
 +    1 1 1         +  1 1 3
 ---------          -------
  1 1 1 0            1 0 2
```

□ product:

```
1011 · 101
      1011
     0000
    1011
   ------
    110111
```

Generalized binary fixed-point and floating-point numbers

0.75

$$0.75 = 1\cdot\frac{1}{2} + 1\cdot\frac{1}{4} = 0.11_2$$

0.7

$$0.7 = 1\cdot\frac{1}{2} + 0\cdot\frac{1}{4} + 1\cdot\frac{1}{8} + 1\cdot\frac{1}{16} + ...$$

the first 64 bits:
0.1011001100110011001100110011001100110011001100110011001100110011

0.7 is a periodic number in the binary system.

floating point variables

0/1 sequences are interpreted as sign (s) , mantissa (m) and exponent (p)

0 0 0 0 0 0 0 1 0 1 0 1 0 1 0 1 1 0 1 0 1 0 0 0 0 1 0 0 0 0 1 0 1

s p m

the resulting number then is $s\cdot m\cdot 2^p$

In the IEEE-754 standard, 127 is added to the exponent, and the leading 1 of the mantissa is not stored. The exponent has 8 bits and the mantissa 23 explicit bits, thus 24 implicit bits.

0 1 0 0 0 0 0 1 1 0 1 0 1 0 1 1 0 1 0 1 0 0 0 0 1 0 0 0 0 1 0 1

s p' m'

-> representation errors in IEEE format is not avoidable
-> x = 0.7; x = 11.0 · x – 7.0; increases the error by a factor of 10

Wrong results in spite of exact computations

Expand

$$expand\left(\frac{x\cdot(x^3+3)}{x\cdot(x+1)}\right);$$

$$\frac{x^3}{x+1}+\frac{3}{x+1} \tag{5}$$

The case

The fibonacci series is defined as follows:
fib(0) = 0, fib(1) = 1 and fib(n+1) = fib(n-1) + fib(n)
We would like to know whether f(n) might be expressible as

$$fib(n) = \frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

We would like to get some information fast and without lots of hand work.
How can we start working at the exercise? How can Maple help us?

Solution:
Relativly soon, it is clear that:

$$\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^0-\left(\frac{1-\sqrt{5}}{2}\right)^0\right)$$

$$0 \tag{6}$$

and

$$\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^1-\left(\frac{1-\sqrt{5}}{2}\right)^1\right)$$

$$1 \tag{7}$$

Additionally, it must be true that

$$\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right)+\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n}-\left(\frac{1-\sqrt{5}}{2}\right)^{n}\right)=\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$$

Some large numbers can quicjkly be tested, the expression may be simpified via the command simlify. An example is 876:

$$simplify\left(\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{876-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{876-1}\right)+\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{876}-\left(\frac{1-\sqrt{5}}{2}\right)^{876}\right)-\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{876+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{876+1}\right)\right)$$

$$0 \tag{8}$$

The procedure becomes by far more tricky, if we want Maple to show equality for general n. Sometimes, it helps to expand the expression.

$$expand\left(\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right)+\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n}-\left(\frac{1-\sqrt{5}}{2}\right)^{n}\right)-\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)\right);$$

$$\frac{1}{5}\frac{\sqrt{5}\left(\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)^{n}}{\frac{1}{2}+\frac{1}{2}\sqrt{5}}-\frac{1}{5}\frac{\sqrt{5}\left(\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)^{n}}{\frac{1}{2}-\frac{1}{2}\sqrt{5}}+\frac{1}{10}\sqrt{5}\left(\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)^{n}-\frac{1}{10}\sqrt{5}\left(\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)^{n}-\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)^{n}-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)^{n} \tag{9}$$

$$g := simplify\left(expand\left(\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right)+\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n}-\left(\frac{1-\sqrt{5}}{2}\right)^{n}\right)-\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)\right)\right);$$

$$0 \tag{10}$$

$$h := simplify\left(\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right)+\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n}-\left(\frac{1-\sqrt{5}}{2}\right)^{n}\right)-\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)\right);$$

$$\frac{1}{10}\sqrt{5}\left(\left(\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)^{n}+\sqrt{5}\left(\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)^{n}-\left(\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)^{n}+\sqrt{5}\left(\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)^{n}-2\left(\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)^{n+1}+2\left(\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)^{n+1}\right) \tag{11}$$

$$is\left(simplify\left(\left(\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right)+\frac{1}{\sqrt{5}}\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n}\right.\right.\right.\right.$$

$$- \left(\frac{1 - \sqrt{5}}{2} \right)^n \Bigg) \Bigg) - \left(\frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} \right) \right) = 0 \Bigg);$$

$$\mathit{false} \tag{12}$$

>

>

>

> restart; sff := [seq(0, i = 0 ..99)] :

> sff[0] := 0; sff[1] := 1;

Error, out of bound assignment to a list

$$sff_1 := 1 \tag{13}$$

> sff[0 + 1] := 0; sff[1 + 1] := 1;

$$sff_1 := 0$$

$$sff_2 := 1 \tag{14}$$

> sff[2 + 1] := sff[0 + 1] + sff[1 + 1];

$$sff_3 := 1 \tag{15}$$

```
> for i from 3 to 99 do
    sff[i + 1] := sff[i − 1 + 1] + sff[i − 2 + 1];
    if i = 97 then print(sff[i + 1]) fi;
  end do:
```

$$83621143489848422977 \tag{16}$$

> restart; sff := Array(0 ..10000, fill = 0) :

>

> sff[0] := 0; sff[1] := 1;

$$sff_0 := 0$$

$$sff_1 := 1 \tag{17}$$

> sff[2] := sff[0] + sff[1];

$$sff_2 := 1 \tag{18}$$

```
> for i from 3 to 10000 do
    sff[i] := sff[i − 1] + sff[i − 2];
    if i = 97 then print(sff[i]) fi;
  end do:
```

$$83621143489848422977 \tag{19}$$

> sff1 := sff[0]; sff2 := sff[1]; sff3 := sff[2];

$$sff1 := 0$$

$$sff2 := 1$$

$$sff3 := 1 \tag{20}$$

```
> for i from 3 to 10000 do
    sff4 := sff3 + sff2;
    sff2 := sff3 : sff3 := sff4;
    if i = 97 then print(sff3) fi;
```

```
end do:
```

$$83621143489848422977 \tag{21}$$