Mathematical variables, parameters and placeholders

Equations and linear equation systems

e.g.: $2 \cdot x^2 + 5 \cdot x - 2 = 0$ is equivalent to $x^2 + \frac{5}{2} \cdot x - 1 = 0$.

Application of pq-formula results in $x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$.

With $p = \frac{5}{2}$ and $q = -1$, we get

$$x_1 = -\frac{\frac{5}{2}}{2} + \sqrt{\left(\frac{\frac{5}{2}}{2}\right)^2 + 1}; \text{ and } x_2 = -\frac{\frac{5}{2}}{2} - \sqrt{\left(\frac{\frac{5}{2}}{2}\right)^2 + 1};$$

$$-\frac{5}{4} - \frac{1}{4}\sqrt{41} \quad (1)$$

$$-\frac{5}{4} + \frac{1}{4}\sqrt{41} \quad (2)$$

'Nice to have' is something re-usable.
##show

$pq := \textbf{proc}(p, q)$

$\quad \textbf{return } -\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 - q}, -\frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 - q}$; *#returns 2 comma-separated expressions*

end proc;

```
proc(p, q)                                                         (3)
    return -1/2*p + sqrt(1/4*p^2 - q), -1/2*p - sqrt(1/4*p^2 - q)
end proc
```

$pq\left(\frac{5}{2}, -1\right)$;

$$-\frac{5}{4} + \frac{1}{4}\sqrt{41}, -\frac{5}{4} - \frac{1}{4}\sqrt{41} \quad (4)$$

##re-usable, time-dependent variables. Here place-holder a:

$a := 2;$

$$2 \quad (5)$$

$a := x^2 + 1;$

$$x^2 + 1 \quad (6)$$

$a := a + 1;$

$$x^2 + 2 \quad (7)$$

Simplification and Evaluation (numeric vs. symbolic, algorithmic vs. heuristic)

restart;

Numbers: $\frac{18}{6}$, $\frac{18.01}{6.03}$, $\sqrt{2}$, $6\cdot\sqrt{2}$, $\sqrt{2}^2$

$$3, 2.986733002, \sqrt{2}, 6\sqrt{2}, 2 \quad \textbf{(8)}$$

Symbolic expressions: $\frac{a\cdot(b+1)}{a}$;

$$b+1 \quad \textbf{(9)}$$

$factor\left(x^2+\frac{2\cdot p}{2}\cdot x+\left(\frac{p}{2}\right)^2\right)$; *#symbolic, exact, algorithmic*

$$\frac{1}{4}\,(p+2\,x)^2 \quad \textbf{(10)}$$

$x := 2;$

$$2 \quad \textbf{(11)}$$

> $\mathrm{sqrt}(a^2)$;

$$\sqrt{a^2} \quad \textbf{(12)}$$

> $simplify(\mathrm{sqrt}(a^2))$;

$$\mathrm{csgn}(a)\,a \quad \textbf{(13)}$$

> $\mathrm{sqrt}(a^2)$ assuming $a < 0$;

$$-a \quad \textbf{(14)}$$

> $simplify(\mathrm{sqrt}(a^2))$ assuming $a :: real, a > 0$;

$$a \quad \textbf{(15)}$$

> $simplify(\mathrm{sqrt}(a^2))$ assuming $a :: real$;

$$|a| \quad \textbf{(16)}$$

>

The following expression leads to a surprising answer. Why? Thus: be careful!

> $simplify(\sin(x)^2\cdot x^4+\cos(x)^2\cdot x^4)$;

> $simplify(\sin(y)^2\cdot y^4+\cos(y)^2\cdot y^4)$;

> *restart*;

> $simplify(\sin(x)^2\cdot x^4+\cos(x)^2\cdot x^4)$;

Complex Numbers

- a complex number z is of the form a + bi, with i^2= -1 and a,b ∈ℝ. a = Re(z) is the real part of z and b=Im(z) is the imaginary part of z. An equivalent definition is via a two dimensional vector (a,b).
- two complex numbers are equal if and only if their real parts and their imaginary parts are equal

- Complex numbers are added, subtracted, multiplied, and divided by formally applying the associative, commutative and distributive laws of algebra, together with the equation i^2 = -1.
  - Addition : (a+bi) + (c+di) = (a+c) + (b+d)i [in vector notation: (a,b) + (c,d) = (a+c, b+d)]
  - Substraction : (a+bi) - (c+di) = (a-c) + (b-d)i
  - Multiplication: $(a+bi)\cdot(c+di) = (ac-bd) + (bc+ad)i$
  - Division : $\frac{a+bi}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$, with c or d not equal to 0

- with the given definitions of addition, substraction, multiplication, division, and
  - the additive identity (zero-element) 0 + 0i,
  - the multiplicative identity (one-element) 1 + 0i,
  - the addidive inverse of a number a + bi: -a - bi, and
  - the multiplicative inverse of a + bi: $\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$,

  the complex numbers ℂ are a *field* (dt: Körper)

Numeric complex computations

> $\frac{(3+3\cdot I)}{(2+6\cdot I)}$;

$$\frac{3}{5} - \frac{3}{10}\,\mathrm{I} \qquad \textbf{(17)}$$

> $\left(\frac{3}{3^2+5^2} + \frac{(-5)}{3^2+5^2}\cdot I\right)\cdot(3+5\cdot I)$;

$$1 \qquad \textbf{(18)}$$

Symbolic complex computations
Simplifying an expression

> *restart*;

> $\left(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}\cdot I\right)\cdot(a+b\cdot I)$ assuming $a > 0$;

$$\left(\frac{a}{a^2+b^2} - \frac{\mathrm{I}\,b}{a^2+b^2}\right)(a+\mathrm{I}\,b) \qquad \textbf{(19)}$$

>

> *simplify*(%);

$$1 \qquad \textbf{(20)}$$

> $solve(x^2+1=0);$

$$I, -I \tag{21}$$

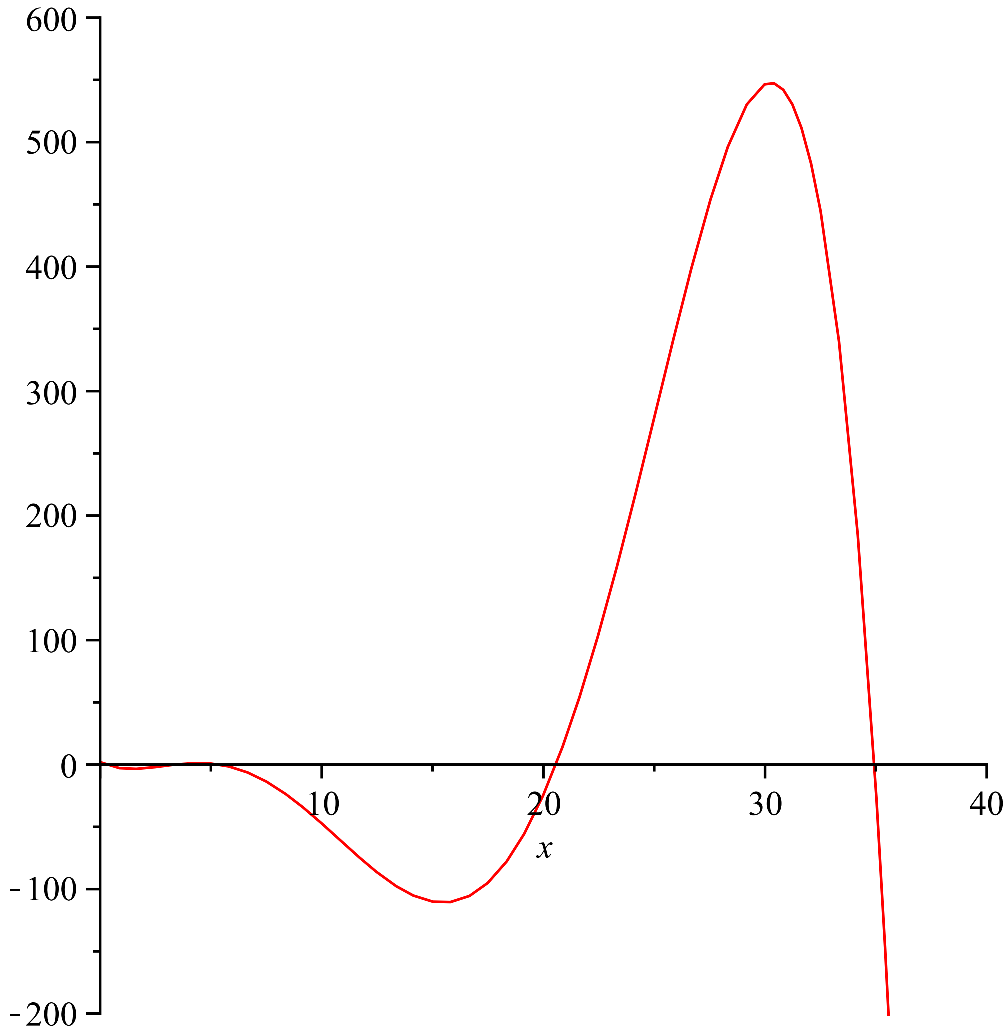
Programming with proc, for and if

Find all local maxima of a polynomial f

$f := x \rightarrow -\frac{683161}{1133371470}x^5 + \frac{11752043}{302232392}x^4 - \frac{112862553}{151116196}x^3 + \frac{198166575}{43176056}x^2 - \frac{416037877}{46260060}x + 2;$

$$x \rightarrow -\frac{683161}{1133371470}x^5 + \frac{11752043}{302232392}x^4 - \frac{112862553}{151116196}x^3 + \frac{198166575}{43176056}x^2 - \frac{416037877}{46260060}x + 2 \tag{22}$$

$plot(f(x), view = [0..40, -200..600], x = 0..40);$



$fsolve(f'(x) = 0, x);$

$$1.431724935, 4.453992057, 15.46691845, 30.25471480 \tag{23}$$

```
evalf(eval(diff(diff(f(x), x), x), x = 1.431724935));
                                3.684775852                                    (24)
evalf(f''(4.453992057)); evalf(f''(15.46691845)); evalf(f''(30.25471480));
                                -2.588142884
                                6.888828816
                                -33.14325482                                   (25)
maxima := proc(f)
            local c, z, el;
            c := 0;
            z := [fsolve(f'(x) = 0, x)];
            for el in z do
              if f''(el) < 0 then c := c + 1;
              end if;
            end do;
            return c;
 end proc;
proc(f)                                                                        (26)
    local c, z, el;
    c := 0;
    z := [fsolve(diff(f(x), x) = 0, x)];
    for el in z do if eval(diff(f(x), x, x), x = el) < 0 then c := c + 1 end if end do;
    return c
end proc
maxima(f);
                                     2                                         (27)

## analyze procedure, list, set, sequence

z := [fsolve(f'(x) = 0, x)];
              [1.431724935, 4.453992057, 15.46691845, 30.25471480]             (28)

for el in z do
  print(el)
end do;
                                1.431724935
                                4.453992057
                                15.46691845
                                30.25471480                                    (29)
## sequences

s := 3, 5, 7;
                                  3, 5, 7                                      (30)
t := 2, 4, 6;
                                  2, 4, 6                                      (31)
t2 := s, t;
```

$$3, 5, 7, 2, 4, 6 \qquad (32)$$

lists

$l1 := [3, 5, 7];$

$$[3, 5, 7] \qquad (33)$$

$l2 := [2, 4, 6];$

$$[2, 4, 6] \qquad (34)$$

$l3 := [l2, l1];$

$$[[2, 4, 6], [3, 5, 7]] \qquad (35)$$

sets

$s1 := \{1, 2, 3, 4\}; s2 := \{3, 4, 5\};$

$$\{1, 2, 3, 4\}$$
$$\{3, 4, 5\} \qquad (36)$$

$s3 := s1$ **union** $s2;$

$$\{1, 2, 3, 4, 5\} \qquad (37)$$

Syntactical description of control structures (homework):

Flow Control (if, for, while, ...)

if <conditional expression> **then** <statement sequence>
| **elif** <conditional expression> **then** <statement sequence> |
| **else** <statement sequence> |
end if

(Note: Phrases located between | | are optional.)

The **for ...while ... do** loop
| **for** <name> | | **from** <expr> | | **by** <expr> | | **to** <expr> | | **while** <expr> |
do <statement sequence> **end do**;

OR

| **for** <name> | | **in** <expr> | | **while** <expr> |
do <statement sequence> **end do**;

(**Note:** Clauses shown between | | above are optional, and can appear in any order, except that the for clause, if used, must appear first.)

Procedures

Flow control constructions, simple commands and comparison operators can be bound together; in a so called
procedure. The simplest possible procedure looks as follow.

proc(parameter sequence)
 statements;
end proc: