Analysis III – Complex Analysis
8. Exercise Sheet

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Groupwork

Exercise G1 (A strange Laurent series expansion)

Consider the following Laurent series expansion of the zero function:

$$\begin{aligned} 0 &= \frac{1}{z-1} + \frac{1}{1-z} = \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} + \frac{1}{1-z} \\ &= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} z^n = \sum_{n=-\infty}^{\infty} z^n. \end{aligned}$$

This contradicts the uniqueness of the Laurent series expansion, doesn't it?

Exercise G2 (Some Laurent series expansions)

Consider the holomorphic function $f : \mathbb{C} \setminus \{1,3\} \to \mathbb{C}$, $f(z) = \frac{2}{z^2-4z+3}$. Use the partial fraction decomposition

$$f(z) = \frac{1}{1-z} + \frac{1}{z-3}$$

to expand f on the following annuli into a Laurent series in $z_0 = 0$:

$$R_1 := \{z \in \mathbb{C} :\ 0 < |z| < 1\}, \quad R_2 := \{z \in \mathbb{C} :\ 1 < |z| < 3\}, \quad R_3 := \{z \in \mathbb{C} :\ 3 < |z| < 42\}.$$

Exercise G3 (On residues of holomorphic functions)

Let $f : \Omega \to \mathbb{C}$ be a holomorphic function and assume there is an $r > 0$ such that $K_{r,0}(z_0) \subseteq \Omega$ where $K_{r,0}(z_0) := \{z \in \mathbb{C} :\ 0 < |z - z_0| < r\}$.
Remember that the *residue of f in z_0* is defined by $\operatorname{Res}(f, z_0) := a_{-1}$ where $\sum_{k=-\infty}^{\infty} a_k \cdot z^k$ is the Laurent series expansion of f converging in $K_{r,0}(z_0)$ to f.

(a) Let $n \in \mathbb{N}$ be a natural number such that $z \to (z - z_0)^n \cdot f(z)$ has a holomorphic extension on $\Omega \cup \{z_0\}$ (e. g. if f has in z_0 a pole of order at most n). Show:

$$\operatorname{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left((z - z_0)^n \cdot f(z) \right).$$

(b) Let $g, h : \Omega \cup \{z_0\} \to \mathbb{C}$ be holomorphic. Assume that h has in z_0 a zero of order 1 and set $f(z) := \frac{g(z)}{h(z)}$. Show:

$$\operatorname{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}.$$

(c) Calculate the following integrals:

$$\text{(i)} \int_{C_1(0)} \frac{e^z}{\sin(z)} dz, \qquad \text{(ii)} \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx, \qquad \int_{C_1(0)} \frac{1}{|z|} dz.$$

Exercise G4 (Singularities)

If $f : \Omega \to \mathbb{C}$ is holomorphic we call a point $z_0 \in \mathbb{C}$ an *isolated singularity* of f if $z_0 \notin \Omega$ and $K_{r,0}(z_0) = \{z \in \mathbb{C} : 0 < |z - z_0| < r\} \subseteq \Omega$ for some $r > 0$. We want to discuss three types of singularities:

An isolated singularity z_0 of f is called a *removable singularity* if f has a holomorphic extension on $\Omega \cup \{z_0\}$.
An isolated singularity z_0 of f is called a *pole* if z_0 is not a removable singularity of f and there exists a $n > 0$ such that $z \to (z - z_0)^n \cdot f(z)$ has a removable singularity in z_0. The smallest number $n \in \mathbb{N}$ with this property is called the *order* of the pole.
An isolated singularity z_0 of f is called an *essential singularity* if z_0 is neither a removable singularity nor a pole.

(a) Find an example for each kind of an isolated singularity.

(b) Show: Let $f : \Omega \to \mathbb{C}$ be holomorphic and z_0 be an isolated singularity. Then there are equivalent:
   (i) The singularity z_0 is removable.
   (ii) There is a power series expansion of f in z_0 converging on $K_r(z_0)$.

(c) Show: Let $f : \Omega \to \mathbb{C}$ be holomorphic and $z_0 \in \Omega$ be an isolated singularity. Then there are equivalent:
   (i) The singularity z_0 is a pole.
   (ii) The principal part of the Laurent series expansion of f in z_0 on $K_{r,0}(z_0)$ is not trivial and all but finitely many coefficients vanish.

(d) Consider the holomorphic functions

$$f(z) = \frac{\sin(z)}{z}, \quad g(z) = \sin\left(\frac{1}{z}\right), \quad h(z) = \frac{1}{\sin(z)}$$

on there natural domains. Each of these functions have in $z_0 = 0$ an isolated singularity. Classify the isolated singularities. **Hint:** You could use the result of (f).

(e) In excercise G2 you determined some Laurent series in z_0 with infinite principal part. Does this mean the function f has an essential singularity in $z_0 = 0$?

(f) Let $f : \Omega \to \mathbb{C}$ be holomorphic and $z_0 \in \Omega$ be a pole of f. Then $\lim_{z \to z_0} |f(z_0)| = \infty$.

(g) The function $f(z) := \exp\left(-\frac{1}{z^2}\right)$ has an essential singularity in $z_0 = 0$. Show: For each $\omega \in \mathbb{C}$ there is a null sequence $(z_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} f(z_n) = \omega$.

The phenomenon in (g) is typical for essential singularities cf. the Casorati-Weierstrass Theorem or – a much stronger fact – the Big Picard Theorem in the literature.