Analysis III – Complex Analysis
7. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

Department of Mathematics
Prof. Dr. Burkhard Kümmerer
Andreas Gärtner
Walter Reußwig

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Groupwork

Exercise G1 (The Fundamental Theorem of Algebra)

Use Liouville's Theorem to prove the Fundamental Theorem of Algebra: Every polynomial $p : \mathbb{C} \to \mathbb{C}$ which has no root is constant.
Hint: Consider the rational function $f(z) = \frac{1}{p(z)}$. Show this function has to be bounded if p has no roots.

Exercise G2 (Complex powers of complex numbers)
Let $z, \omega \in \mathbb{C} \setminus \{0\}$ be complex numbers and let $l : \Omega \to \mathbb{C}$ be a logarithm with $z \in \Omega$. We define

$$z^\omega := \exp(l(z) \cdot \omega).$$

Of course this definition depends on the logarithm l. For simplicity we shall choose the principal value Log of the logarithm, i. e. the logarithm function on $\Omega := \mathbb{C} \setminus]-\infty, 0[$ with $\mathrm{Log}(1) = 0$.

(a) Determine i^i.

(b) One might expect the identities

$$\begin{aligned} z^{\omega_1+\omega_2} &= z^{\omega_1} \cdot z^{\omega_2}, \\ z_1^\omega \cdot z_2^\omega &= (z_1 \cdot z_2)^\omega, \\ (z^{\omega_1})^{\omega_2} &= z^{\omega_1 \cdot \omega_2}. \end{aligned}$$

Discuss this.

Exercise G3 (The complex sine function)

(a) Determine every zero of the complex sine, i. e. every $z \in \mathbb{C}$ with $\sin(z) = 0$.

(b) Show: The function $f(z) := \frac{\sin(z)}{z}$ is holomorphic on $\Omega := \mathbb{C} \setminus \{0\}$ and has a unique holomorphic extension to an entire function.

(c) Determine the integrals

$$\text{(i)} \int_{C_1(0)} \frac{z}{\sin(z)} dz \quad \text{and} \quad \text{(ii)} \int_{C_1(0)} \frac{1}{\sin(z)} dz.$$

Exercise G4 (Cauchy Integral Formula)

Determine the integrals

$$(i)\ \int_{C_2(i)} \frac{1}{z^2+4}dz, \qquad (ii)\ \int_{C_2(i)} \frac{1}{(z^2+4)^2}dz.$$

Homework

Exercise H1 (A generalisation of Liouville's theorem) (1 point)

Let $f : \mathbb{C} \to \mathbb{C}$ holomorphic. Further assume there are constants $a, b \in]0, \infty[$ and a natural number $n \in \mathbb{N}$ with $|f(z)| \leq a \cdot |z|^n + b$ for all $z \in \mathbb{C}$. Show that f is a polynomial with $\deg(f) \leq n$.

Exercise H2 (Power Series) (1 point)

(a) Let $f : \Omega \to \mathbb{C}$ a holomorphic function and $K_r(z_0) \subseteq \Omega$ for some $r > 0$. If f is unbounded on $K_r(z_0)$ then the power series expansion of f in z_0 has radius of convergence r.

(b) Determine the radius of convergence for the power series expansion in $z_0 = 0$ of the following functions

$$(i)\quad f(z) = \frac{1}{z+i}, \qquad (ii)\quad g(z) = \frac{1}{z^2+z+1}, \qquad (iii)\quad g(z) = \frac{1}{\cos(z)}.$$

Exercise H3 (The biholomorphic maps of the open unit disk) (1 point)

In this excercise we discuss the biholomorphic transformations of the open unit disk $\mathbb{D}$, i. e. the set

$$\mathrm{Aut}(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{D},\ f \text{ is holomorphic, bijective and its inverse is again holomorphic}\}.$$

Obviously this set forms a subgroup of the group of all bijections of $\mathbb{D}$. We call an element $f \in \mathrm{Aut}(\mathbb{D})$ an *automorphism of* $\mathbb{D}$.

To understand this group, we first prove Schwarz's Lemma. This will help us to determine the automorphisms which leaves the point $0 \in \mathbb{D}$ fix. Then we classify the automorphisms of $\mathbb{D}$.

(a) Prove Schwarz's Lemma: If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic with $f(0) = 0$ then we have for all $z \in \mathbb{D}$ the estimation $|f(z)| \leq |z|$.
Further if there exists a $z_0 \in \mathbb{D}$ with $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$ then $f(z) = \lambda \cdot z$ for some $\lambda \in \mathbb{T}$, i. e. f is a rotation.
Hint: Consider the function $g(z) := \frac{f(z)}{z}$ and use the maximum principle.

(b) Show that every automorphism $f \in \mathrm{Aut}(\mathbb{D})$ with $f(0) = 0$ is a rotation.

(c) Show that every element of the set

$$J := \left\{ f(z) = \frac{az+b}{\overline{b}z+\overline{a}} \,\middle|\, a, b \in \mathbb{C} : |a|^2 - |b|^2 = 1 \right\}$$

is an automorphism of $\mathbb{D}$ and show that J is a subgroup of $\mathrm{Aut}(\mathbb{D})$. Further show

$$J = \left\{ f(z) = e^{i\varphi} \cdot \frac{z-\omega}{\overline{\omega} \cdot z - 1} \,\middle|\, \omega \in \mathbb{D},\ 0 \leq \varphi < 2\pi \right\}.$$

(d) Fix $\omega \in \mathbb{D}$. Find an automorphism $f \in J$ with $f(0) = \omega$.

(e) Prove: If $H \subseteq \mathrm{Aut}(\mathbb{D})$ is a subgroup which satisfies

  (i) for every $z, w \in \mathbb{D}$ there is an automorphism $f \in H$ with $f(z) = w$ (H acts transitively on $\mathbb{D}$),

  (ii) there is a point $z \in \mathbb{D}$ such that $f \in \mathrm{Aut}(\mathbb{D})$ with $f(z) = z$ implies $f \in H$ (H contains the stabiliser of some $z \in \mathbb{D}$),

  then $H = \mathrm{Aut}(\mathbb{D})$.

Conclude

$$\begin{aligned}\mathrm{Aut}(\mathbb{D}) &= \left\{ f(z) = \frac{az+b}{\overline{b}z+\overline{a}} \,\middle|\, a,b \in \mathbb{C} : |a|^2 - |b|^2 = 1 \right\} \\ &= \left\{ f(z) = e^{i\varphi} \cdot \frac{z-\omega}{\overline{\omega} \cdot z - 1} \,\middle|\, \omega \in \mathbb{D},\ 0 \leq \varphi < 2\pi \right\}.\end{aligned}$$

(f) Show: Every $f \in \mathrm{Aut}(\mathbb{D})$ extends to $\overline{\mathbb{D}}$ and maps $\mathbb{T}$ bijective to $\mathbb{T}$.