Analysis III – Complex Analysis
6. Exercise Sheet

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Groupwork

Exercise G1 (Cauchy Integral Formula)

Use the Cauchy Integral Formula to determine the following integrals:

$$(a) \oint_{C_1(i)} \frac{1}{z-i} dz, \qquad (b) \oint_{C_1(i)} \frac{1}{z^2+1} dz, \qquad (c) \oint_{C_{42}(i)} \frac{1}{z^2+1} dz.$$

Hint: To decompose the integral in (c) into elementary circle integrals use the homotopy invariance of the path integral.

Exercise G2 (Radius of convergence)

Consider a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ which has radius of convergence $r > 0$. Show: There is no holomorphic extension of f on $K_R(0)$ for any $R > r$.

Exercise G3 (The Cauchy transform)
Let $f : \mathbb{T} \to \mathbb{C}$ be a continuous function. We define a function $\widehat{f} : \mathbb{D} \to \mathbb{C}$ by

$$\widehat{f}(z) := \frac{1}{2\pi i} \oint_{\mathbb{T}} \frac{f(\omega)}{\omega - z} d\omega.$$

(a) Show that $\widehat{f}$ is holomorphic.

(b) Show: If $f : \Omega \to \mathbb{C}$ with $\overline{\mathbb{D}} \subseteq \Omega$ is holomorphic then $\widehat{f} = f$.

(c) Is it always true that $\lim_{z\to\omega,\ z\in\mathbb{D}} \widehat{f}(z) = f(\omega)$ holds for $\omega \in \mathbb{T}$?

The function $\widehat{f}$ is called the *Cauchy transform* of f.

Homework

Exercise H1 (Conjugation with reflections) (1 point)

Let $L \subseteq \mathbb{C}$ a one dimensional real subspace of $\mathbb{C}$. Further let $\sigma : \mathbb{C} \to \mathbb{C}$ the real linear map with $\sigma(l) = l$ for all $l \in L$ and $\sigma(l^\perp) = -l^\perp$ for all $l \in L^\perp$ where $L^\perp$ is the orthogonal complement of L with resprect to the canonical scalar product on $\mathbb{R}^2$. Of course this means that σ is orthogonal with determinant -1.

(a) Let $\Omega \subseteq \mathbb{C}$ be a domain and $f : \Omega \to \mathbb{C}$ a function. Show that the following statements are equivalent:
  (i) $f : \Omega \to \mathbb{C}$ is holomorphic.
  (ii) $\sigma \circ f \circ \sigma : \sigma(\Omega) \to \mathbb{C}$ is holomorphic.

Now let $\Omega \subseteq \mathbb{C}$ be a domain which is symmetric a to the real axis, i. e. $\Omega = \{\overline{z} : z \in \Omega\}$. We define $f^*(z) := \overline{f(\overline{z})}$. From (a) it follows that f^* is holomorphic if and only if f is holomorphic.

(b) Determine the derivative of f^* directly.

(c) Assume $f(z) = \sum_{k=0}^{\infty} a_k \cdot z^k$ converges on Ω. Determine the power series of f^*. Which holomorphic functions of this form satisfy $f = f^*$?

(d) Show that every holomorphic function on Ω is linear combination of two holomorphic functions g, h on Ω with $g = g^*$ and $h = h^*$.
Hint: To get an idea you could first prove (d) for holomorphic functions given by a power series like in (c).

Exercise H2 (The mean value property) (1 point)

Let $\Omega \subseteq \mathbb{C}$ a simply connected domain.

(a) Show that the following statements are equivalent for a function $u : \Omega \to \mathbb{R}$:
  (i) $\Delta u(z) = 0$ for each $z \in \Omega$ where Δ is the Laplacian if we identify Ω as a subset of $\mathbb{R}^2$.
  (ii) There is a holomorphic function $f : \Omega \to \mathbb{C}$ with $u(z) = \mathrm{Re}(f(z))$.
  We call a function u satisfiing (i) *harmonic on* Ω.

(b) Show the mean value property for harmonic functions: If $u : \Omega \to \mathbb{R}$ is harmonic and $\overline{K_r(z_0)} \subseteq \Omega$ holds, then

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + r \cdot e^{it}) dt.$$

Now let $\Omega \subseteq \mathbb{C}$ be an arbitrary domain.

(c) Let $u : \Omega \to \mathbb{R}$ be a harmonic function. Since Ω need not to be simply connected, we can't conclude that u is the real part of a holomorphic function. Why does u satisfy the mean value property anyway?

Exercise H3 (Real integrals and complex path integrals) (1 point)

In this exercise we want to calculate the integral:

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx.$$

We will see that the complex line integral could be a mighty help for real integration.

(a) Calculate the roots of the polynomial $p(z) = z^4 + 1$. Sketch them into the unit circle and decide which of them lie in the upper half plane $H := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$.

(b) Show that the real integral

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx$$

exists and is finite.

(c) Let $r \in]1, \infty[$ an arbitrary number. Consider the paths

$$\begin{aligned} \gamma_r^{(1)} : [0,1] \to \mathbb{C}, \quad \gamma_r^{(1)}(t) &:= r(2t-1), \\ \gamma_r^{(2)} : [0,1] \to \mathbb{C}, \quad \gamma_r^{(2)}(t) &:= r \cdot e^{i\pi t}. \end{aligned}$$

and set $\gamma_r := \gamma_r^{(1)} + \gamma_r^{(2)}$. Assure yourself that γ_r is a loop in $\mathbb{C}$. Sketch the path γ_r for a suitable choice of $r > 1$ and argue that

$$\int_{\gamma_r} \frac{z^2}{z^4+1} dz$$

is independent of the choosen $r \in]1, \infty[$.

(d) Use the standard estimation to show

$$\lim_{r\to\infty} \int_{\gamma_r^{(2)}} \frac{z^2}{z^4+1} dz = 0. \quad \text{Conclude:} \quad \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx = \int_{\gamma_r} \frac{z^2}{z^4+1} dz$$

for any $r > 1$.

(e) Use the Cauchy Integral Formular and the factorisation of $z^4 + 1$ into linear factors to show

$$\frac{1}{2\pi i} \cdot \oint_{C_1(\xi_1)} \frac{z^2}{z^4+1} dz = \frac{\xi_1^2}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)(\xi_1 - \xi_4)}$$

$$\frac{1}{2\pi i} \cdot \oint_{C_1(\xi_2)} \frac{z^2}{z^4+1} dz = \frac{\xi_2^2}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)(\xi_2 - \xi_4)}$$

where $\xi_k = e^{\frac{\pi i(2k+1)}{4}}$ are the roots of $z^4 + 1$.

(f) Argue

$$\int_{\gamma_r} \frac{z^2}{z^4+1} dz = \oint_{C_1(\xi_1)} \frac{z^2}{z^4+1} dz + \oint_{C_1(\xi_2)} \frac{z^2}{z^4+1} dz$$

for any $r > 1$. Finally determine this integral.