Analysis III – Complex Analysis
5. Exercise Sheet

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Groupwork

Exercise G1 (Standard estimations)

(a) Consider a continuous function $f : [0,1] \to \mathbb{C}$. Show

$$\left| \int_0^1 f(t)dt \right| \leq \int_0^1 \left| f(t) \right| dt.$$

Hint: For each complex number $z \in \mathbb{C}$ there is a complex number $\omega \in \mathbb{C}$ such that $\omega \cdot z = |z|$.

(b) Let $\Omega \subseteq \mathbb{C}$ be a domain and $f : \Omega \to \mathbb{C}$ be a continuous function. Further let $\gamma : [0,1] \to \Omega$ be an arbitrary path. Show the standard estimation for the complex path integral:

$$\left| \int_\gamma f(z)dz \right| \leq \sup_{t\in[0,1]} \{|f(\gamma(t))|\} \cdot L(\gamma) < \infty.$$

Exercise G2 (Locally uniform convergence)

Let $\Omega \subseteq \mathbb{C}$ be a domain. If $f : \Omega \to \mathbb{C}$ is an arbitrary function then we denote by f^K for $K \subseteq \Omega$ the restriction of f on K, i. e. $f^K : K \to \mathbb{C}$, $f^K(z) := f(z)$.
Let $f_n : \Omega \to \mathbb{C}$ be a function for each $n \in \mathbb{N}$. We say the sequence $(f_n)_{n\in\mathbb{N}}$ converges **locally uniformly** to a function $f : \Omega \to \mathbb{C}$ if for each compact subset $K \subseteq \Omega$ the sequence $(f_n^K)_{n\in\mathbb{N}}$ converges uniformly to f^K.

(a) Show: If $(f_n)_{n\in\mathbb{N}}$ converges locally uniformly to f then it converges pointwise to f.

(b) Show: If $(f_n)_{n\in\mathbb{N}}$ converges locally uniformly to f and if the function f_n is continuous for each $n \in \mathbb{N}$ then f is continuous.

(c) Give an example of a locally uniformly convergent sequence which is not uniformly convergent.

(d) Show: If $(f_n)_{n\in\mathbb{N}}$ converges locally uniformly to f then for every path $\gamma : [0,1] \to \Omega$ we have

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz.$$

(e) Consider the domain $\Omega := \mathbb{C} \setminus \{0\}$ and the following rational functions:

$$f_n(z) := \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{1}{z^k}.$$

Show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges locally uniformly to $f(z) = e^{\frac{1}{z}}$ and determine the path integral $\oint_{K_1(0)} e^{\frac{1}{z}} dz$.

Exercise G3 (Radius of convergence)

Let $(a_n)_{n\in\mathbb{N}}$ be a monotonically decreasing null sequence and define the power series

$$f(z) := \sum_{n=0}^{\infty} a_n z^n.$$

(a) Show that the radius of convergence of f is at least 1.

(b) Show that for each $z \in \mathbb{T} \setminus \{1\} = \{z \in \mathbb{C} \setminus \{1\} : |z| = 1\}$ the series converges.

Hint: Use $\sum_{k=m}^{n} a_k z^k = \frac{1}{1-z} \cdot \left((1-z) \cdot \sum_{k=m}^{n} a_k z^k\right)$ and estimate the second term.

Homework

Exercise H1 (Real parts of complex differentiable functions) (1 point)

Consider the polynomial $p(x,y) := x^2 + 2axy + by^2$ where $a,b \in \mathbb{R}$ are parameters. Decide for which choices $a,b \in \mathbb{R}$ the polynomial is the real part of a complex differentiable function $f : \mathbb{C} \to \mathbb{C}$, i. e.

$$p(x,y) = \mathrm{Re} f(x+iy).$$

On the other side if $p(x,y) = \mathrm{Re} f(x+iy) = \mathrm{Re} g(x+iy)$ for complex differentiable functions $f,g : \mathbb{C} \to \mathbb{C}$ what can you say about the relationship of f and g?

Exercise H2 (The Gamma function) (1 point)

There are many connections between number theory and complex analysis. In this excercise we construct a complex differentiable function $\Gamma : \Omega \to \mathbb{C}$ which interpolates the factorials. We use the following result:

Theorem: Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f : \Omega \times]0,\infty[\to \mathbb{C}$ a function satisfying the following three conditions:

(i) For every $z \in \Omega$ one has $\int_0^\infty |f(z,t)| dt < \infty$.

(ii) For every $t \in]0,\infty[$ the function $z \to f(z,t)$ is complex differentiable.

(iii) For every compact disk $K = K_r(z_0) \subseteq \Omega$ there is a positive function $g_K :]0,\infty[\to \mathbb{R}_0^+$ with $|f(z,t)| \leq g_K(t)$ for all $t \in]0,\infty[$ and all $z \in K$ and one has

$$\int_0^\infty g_K(t) dt < \infty.$$

Then the function $F : \Omega \to \mathbb{C}$,

$$F(z) := \int_0^\infty f(z,t) dt$$

is complex differentiable and its derivatives are given by

$$F^{(n)}(z) = \int_0^\infty \frac{\partial^n}{\partial z^n} f(z,t) dt.$$

Now we use the domain $\Omega_+ := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ which is the open right half complex plane.

(a) Show that the following statements are equivalent for $\Omega = \Omega_+$:

  (iii) For every compact disk $K = K_r(z_0) \subseteq \Omega_+$ there is a positive function $g_K :]0,\infty[\to \mathbb{R}_0^+$ with $|f(z,t)| \le g_K(t)$ for all $t \in]0,\infty[$ and all $z \in K$ and one has

$$\int_0^\infty g_K(t)dt < \infty.$$

  (iii') For every compact rectangular $K = [a,b] \times i \cdot [c,d] \subseteq \Omega_+$ there is a positive function $g_K :]0,\infty[\to \mathbb{R}_0^+$ with $|f(z,t)| \le g_K(t)$ for all $t \in]0,\infty[$ and all $z \in K$ and one has

$$\int_0^\infty g_K(t)dt < \infty.$$

(b) Show that

$$\Gamma_+(z) := \int_0^\infty t^{(z-1)} \cdot e^{-t} dt := \int_0^\infty e^{\ln(t)\cdot(z-1)} \cdot e^{-t} dt$$

defines a complex differentiable function on Ω_+.

(c) Show the following formulas:

$$\begin{aligned} \Gamma_+(1) &= 1, \\ \Gamma_+(z+1) &= z \cdot \Gamma_+(z) \quad \text{for all } z \in \Omega_+. \end{aligned}$$

Conclude $\Gamma_+(n+1) = n!$ which means the function Γ_+ is indeed a complex differentiable interpolation of the factorials on the right complex half plane.

(d) Show that the function Γ_+ is bounded on the strip $S := \{z \in \mathbb{C} : 1 \le \mathrm{Re}(z) \le 2\}$.

(e) Define $\Omega_0 := \Omega_+$ and $\Omega_{n+1} := \{z \in \mathbb{C} : \mathrm{Re}(z) > -n\} \setminus \{k \in \mathbb{Z} : k \le 0\}$. Further define

$$\begin{aligned} f_0 : \Omega_0 \to \mathbb{C}, \quad f_0(z) &:= \Gamma_+(z), \\ f_{n+1} : \Omega_{n+1} \to \mathbb{C}, \quad f_{n+1}(z) &:= \frac{f_n(z+1)}{z}. \end{aligned}$$

Show: For all $n \in \mathbb{N}$ the function f_n is complex differentiable and agrees on Ω_k with f_k for all $k \le n$. Thus there is a complex differentiable function $\Gamma : \Omega := \mathbb{C} \setminus \{k \in \mathbb{Z} : k \le 0\} \to \mathbb{C}$ with $\Gamma(z) = f_n(z)$ for every $n \in \mathbb{N}$ and all $z \in \Omega_n$.

(f) Show: For each $n \in \mathbb{N}$ one has $\lim_{(z \to -n)} (z+n)\Gamma(z) = \frac{(-1)^n}{n!}$.

For completeness: The Theorem of H. Wieland states the following: Let $\Omega \subseteq \mathbb{C}$ be a domain such that Ω contains the vertical strip S. Then for any function $f : \Omega \to \mathbb{C}$ with

(1) The function f is bounded on S,

(2) The function f satisfies $f(z+1) = z \cdot f(z)$ for all $z \in \Omega$,

one has $f(z) = f(1) \cdot \Gamma(z)$ for all $z \in \Omega$, i. e. the conditions (1) and (2) characterise the Γ-function up to a multiplicative constant.