Analysis III – Complex Analysis
4. Exercise Sheet

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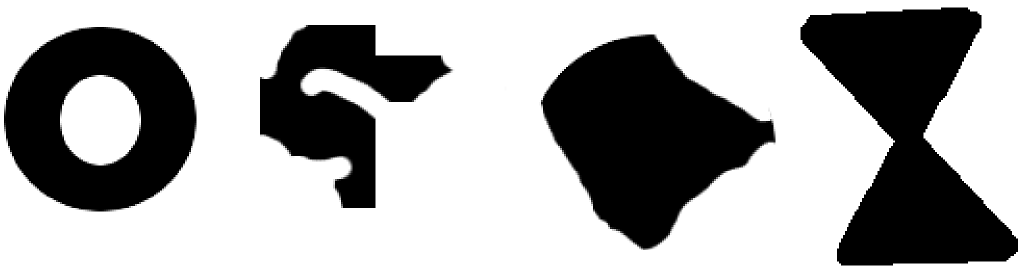
Groupwork

Exercise G1 (Star shaped sets)

(a) Decide which of the scetched subsets of $\mathbb{R}^2$ are star shaped:



a) b) c) d)

(b) Let $X_1 \subseteq \mathbb{R}^n$ and $X_2 \subseteq \mathbb{R}^n$ be two star shaped subsets. Which of the sets $X_1 \cap X_2$, $X_1 \cup X_2$ or $X_1 \times X_2$ are star shaped? Justify your claims.

Remark: You should not use more than 15 minutes for this excercise.

Exercise G2 (Simple connected sets)

(a) Let (X, d_1) and (Y, d_2) be metric spaces and $\varphi : X \to Y$ a homeomorphism. Show: If $A \subseteq X$ is simply connected, then $\varphi(A) \subseteq Y$ is simply connected.

(b) Let $0 < r < R$ be real numbers. Scetch the set

$$K_{r,R} := \{x \in \mathbb{R}^2 : \ r < \|x\| < R\} \setminus \{(0, y) \in \mathbb{R}^2 : \ y \leq 0\}$$

for a suitable choice of r and R. Use polar coordinates to prove that $K_{r,R}$ is simply connected.

(c) Show: The sets $\mathbb{R}^2 \setminus \{0\}$ and $\mathbb{S}^1 := \{x \in \mathbb{R}^2 : \ \|x\| = 1\}$ are not simply connected.

Hint: For (c) consider a useful vector field and use the homotopy invariance of the path integral.

Notes on Homotopy

Let (X,d) be a pathwise connected metric space. For $x \in X$ we use the notation γ_x for the path $\gamma_x : [0,1] \to X$, $\gamma_x(t) = x$ for all $t \in [0,1]$. If γ_1 and γ_2 are paths in X which are homotopic we write $H : \gamma_1 \to \gamma_2$ if H is a homotopy $H : [0,1] \times [0,1] \to X$ for γ_1 and γ_2, i. e. $H(0,t) = \gamma_1(t)$ and $H(1,t) = \gamma_2(t)$ for all $t \in [0,1]$.
Further you can use the theorems of the lectures although a homotopy H is usually not continuously differentiable in this excercise sheet: Treat continuous homotopies like $\mathscr{C}^2$-homotopies. Especially you can use the homotopy invariance of the path integral.

Exercise G3 (Homotopy)

(a) Let (X,d) be a pathwise connected metric space and let $x, y \in X$ be arbitrary points. Show that x-y-homotopy defines an equivalence relation on the set $\Gamma(X,x,y)$ of all paths starting in x and ending in y.

(b) Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Show that the following conditions are equivalent:
   (i) The set Ω is connected.
   (ii) For every $x, y \in \Omega$ the path γ_x is homotopic to γ_y.

Exercise G4 (Homotopy classes of loops on the circle)
We consider the set

$$\pi_1(\mathbb{T},1) := \{[\gamma]_{\simeq} : \ \gamma : [0,1] \to \mathbb{T} \text{ is a continuous path with } \gamma(0) = \gamma(1) = 1\}$$

where $\gamma_1 \simeq \gamma_2$ if and only if there is a homotopy $H : \gamma_1 \to \gamma_2$ with $H(s,0) = H(s,1) = 1$ for all $s \in [0,1]$. Prove that the set $\pi_1(\mathbb{T},1)$ has infinitely many elements.
We know, e. g. from Analysis II, that the set $\pi_1(\mathbb{T},1)$ forms a group with the multiplication $[\gamma_1] \cdot [\gamma_2] := [\gamma_1 + \gamma_2]$ and inversion $[\gamma^{-1}] := [-\gamma]$. Find an isomorphic copy of $\mathbb{Z}$ in $\pi_1(\mathbb{T},1)$.
Remark: In fact the group $\pi_1(\mathbb{T},1)$ is isomorphic to $\mathbb{Z}$ but we can't prove this without further analysis, e. g. the analysis of covering spaces.

Homework

Exercise H1 (Complex path integrals) (1 point)

(a) Determine the following complex path integrals where every path is counterclockwise orientated.
   (i) $\int_{|z|=1} e^{-z^2} dz$.
   (ii) $\int_\gamma \overline{z} dz$, where γ describes the triangle with the endpoints $i, 1-i$ and $-1-i$.

(b) From the lectures we know that every complex path integral can be decomposed into two real path integrals

$$\int_\gamma f\, dz = \int_\gamma \omega_1 ds + i \cdot \int_\gamma \omega_2 ds.$$

Determine for $f(z) = \frac{1}{z}$ the vector fields ω_1 and ω_2. Check the integrability conditions for ω_1 and ω_2 and discuss the existence of primitives of these vector fields.

Exercise H2 (Fundamental Theorem of Algebra) (1 point)

For a real number $r \in [0, \infty[$ we consider the path $\alpha_r : [0,1] \to \mathbb{C}$, $\alpha_r(t) := r \cdot e^{2\pi i t}$ and for a natural number $n \in \mathbb{N}$ we consider the path $\beta_n : [0,1] \to \mathbb{T}$, $\beta_n(t) := e^{2\pi i n t}$. Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial of the form

$$p(z) = z^n + \sum_{k=0}^{n-1} a_k \cdot z^k$$

which has no roots, i. e. there is no point $z_0 \in \mathbb{C}$ with $p(z_0) = 0$.

(a) Let $r > 0$ be a fixed positive real number. Show: The loop $\gamma_r(t) := p(\alpha_r(t))$ is homotopic in $\mathbb{C} \setminus \{0\}$ to the loop $\gamma_0(t) := p(\alpha_0(t))$.

(b) Show: There is a real number $r \geq 1$ such that none of the polynomials

$$f_q(z) := z^n + q \cdot \sum_{k=0}^{n-1} a_k z^k, \quad 0 \leq q \leq 1$$

has a root on $r \cdot \mathbb{T} := \{z \in \mathbb{C} : |z| = r\}$.

(c) Use (b) to find a real number $r \geq 1$ and a homotopy $H : \gamma_r \to r^n \cdot \beta_n$ in $\mathbb{C} \setminus \{0\}$ where n is the degree of p.

(d) Show: p has to be a constant polynomial.

(e) Show the Fundamental Theorem of Algebra: Every complex polynomial f which has no root in $\mathbb{C}$ is constant.

The first complete proof of the Fundamental Theorem of Algebra was given by C. F. Gauß in 1799. During his life Gauß gave three further proofs of this theorem and the following idea of a proof is accredited to Gauß, too:

Let p a complex polynomial of degree n. This polynomial maps loops into loops. If one looks on the image of a loop α_r of very small radius r then the image loop lies in a small neighbourhood of a_0, the absolute part of p. If one looks on a loop of very large radius the image loop behaves like $a_n \cdot r^n \cdot \beta_n$ where n is the degree of p and a_n the highest coefficient of p. Especially it winds n times around the origin. If one varies the radius r continuously the image loops varies continuously in the complex plane: The image loops have to hit the origin, elsewhere the origin can't pass from the exterior of the loops of small radius into the interior of the loops of large radius. So p must have a root.

(f*) Explain shortly in which way our proof in this excercise makes this idea precise.

Exercise H3 (Browers fixed point theorem in dimension 2) (1 point)

Remember the unit disk is defined by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

(a) Prove: There is no continuous map $f : \overline{\mathbb{D}} \to \mathbb{T}$ with $f(z) = z$ for every $z \in \mathbb{T}$.
Hint: If one has a contiuous map $f : X \to Y$ the composition of f with a homotopy in X is a homotopy in Y.

(b) Prove Browers fixed point theorem in dimension 2: Every continuous map $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ has at least one fixed point, i. e. there is a point $z_0 \in \overline{\mathbb{D}}$ with $f(z_0) = z_0$.
Hint: For $z \in \overline{\mathbb{D}}$ consider the set $\{f(z) + \lambda \cdot (z - f(z)) : \lambda > 0\} \cap \mathbb{T}$. This set has exactly one element if f has no fixed point. We call the element of this set $h(z)$. You can use without a proof that $h : \overline{\mathbb{D}} \to \mathbb{T}$ is well defined and continuous.
In which way $h(z)$ depends on z and $f(z)$? Scetch it for some example.